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PRIMARY ELEMENTS

OF

PLANE AND SOLID GEOMETRY:

FOR SCHOOLS AND ACADEMIES.

By E. W. EVANS, M. A.,
PROFESSOR OF MATHEMATICS IN MARIETTA COLLEGE.

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PREFACE.

The following concise treatise on Geometry has been prepared chiefly for that class of students in our Public Schools and elsewhere, who can not spare the requisite time for mastering the larger works. It is conceived, also, that it will be found useful as a course of first lessons for those who intend to pursue the study more at length.

The writer has aimed, in general, to select those propositions in both Plane and Solid Geometry which have the most important direct applications; omitting a large number of those which serve chiefly as steps to something following them in an extended course of reasoning. In order to do this without breaking the logical connection, he has often found it necessary to present the demonstrations in a considerably modified form; though, in a majority of cases, they are essentially based on those found in Euclid, Legendre, and other standard works.

In the treatment of parallel lines, important hints have been taken from the work of Professor Peirce,

whose method on this subject is the most simple and logical. Upon the whole, the demonstrations have been put in that form which was thought best suited to the comprehension of beginners having no previous knowledge of mathematics beyond arithmetic.

In the divisions of the work, the writer has aimed to follow the simple and natural classification of the subjects treated. The arrangement adopted has enabled him to set down as simple corollaries many propositions which are usually demonstrated separately. The definitions have been distributed between the various sections, instead of being crowded together at the beginning. After each Section, or Book, there will be found a few practical illustrations and exercises. In a Supplement, some examples have been given of the application of Algebra to Geometry.

The amount of Geometry contained in the work is sufficient to prepare the pupil for the study of Plane Trigonometry and Surveying.

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INTRODUCTION.

SECTION I.—GENERAL DEFINITIONS.

1. GEOMETRY is that branch of mathematics which treats of *magnitudes in space.*

2. A LINE is that magnitude which has length, without breadth or thickness.

3. A SURFACE is that which has length and breadth, without thickness.

4. A SOLID is that which has length, breadth, and thickness.

5. A POINT has position without extension.

6. A STRAIGHT LINE is one which has the same direction through all its consecutive parts.

7. Of BENT LINES, that which is composed of straight lines is called a *broken line;* that of which no part is straight is called a *curve.*

8. A PLANE is a surface in which if any two points whatever be taken, the straight line joining them will lie wholly in that surface.

9. MAGNITUDES of EXTENSION are lines, surfaces, and solids. The relative positions of these give rise to magnitudes of direction, or *angles,* which will be defined in their proper connections.

SEC. II.—OF TERMS AND SIGNS.

1. An AXIOM is a self-evident truth.

2. A THEOREM is a statement of a truth which requires to be proved, or *demonstrated.*

3. A PROBLEM is a statement of something required to be done, or *solved.*

4. The term PROPOSITION may be applied either to a theorem or to a problem.

5. A COROLLARY is an obvious consequence from something that precedes.

6. A SCHOLIUM is some remark relating to what precedes.

7. The term HYPOTHESIS denotes the supposition made, or the conditions given, in any proposition.

8. The term INFINITE, as used in geometry, means beyond measure; that is, either absolutely beyond limits, or beyond all appreciable limits.

9. A RATIO is the relation which one quantity bears to another, as equal to it, greater, or less. The value of the ratio is the quotient arising from dividing one of the quantities by the other. If A and B represent any two quantities, the ratio of A to B may be written either $A : B$, or $\frac{A}{B}$.

10. A PROPORTION is an equality of ratios. Thus, if the ratio of A to B be equal to the ratio of C to D, those two ratios will constitute a proportion, which may be written $A : B :: C : D$.

11. The sign $=$ denotes the equality of two quantities between which it is placed. Hence,

A proportion may also be written $\frac{A}{B} = \frac{C}{D}$.

12. The sign $+$ denotes addition. Thus, $A+B$ means the sum of A and B, and is read A plus B.

13. The sign $-$ denotes subtraction. Thus, $A-B$ means A diminished by B, and is read A minus B.

14. The sign $\times$ denotes multiplication. Thus, $A\times B$ means A multiplied by B. Sometimes, however, this sign is omitted, especially if one of the factors be a figure. Thus, $2B$ means twice B.

15. The sign $\div$ denotes division. Thus, $A\div B$ means A divided by B. It is equivalent to $\frac{A}{B}$.

16. The parenthesis ( ) denotes that the quantities inclosed within it are to be subjected to the same operation. Thus, $A\times(B+C)$ means A multiplied into the sum of B and C.

17. The square of any quantity, as A, is written A^2; its cube is written A^3.

18. The square root of any quantity, as $A+B$, is written $\sqrt{A+B}$.

SEC. III.—AXIOMS.

There are certain axioms, relating to quantity in general, which lie at the foundation of all branches of mathematics. Others relate particularly to magnitudes in space. Some of those axioms, of both classes, which are most frequently applied in geometry, are stated here.

1. The whole is greater than any of its parts.

2. The whole is equal to the sum of all its parts.

3. Things which are equal to the same thing are equal to each other.

4. If equals be added to equals, the sums will be equal.

5. If equals be taken from equals, the remainders will be equal.

6. If equals be added to unequals, the sums will be unequal.

7. If equals be taken from unequals, the remainders will be unequal.

8. Doubles of the same thing are equal to each other.

9. Halves of equal things are equal to each other.

10. From one point to another only one straight line can be drawn; and that is the shortest line between them.

11. Two magnitudes are equal if, when one is applied to the other, they will exactly coincide.

PLANE GEOMETRY.

BOOK I.

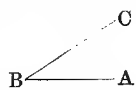
RATIOS OF MAGNITUDES LYING IN THE SAME PLANE.

SEC. IV.—STRAIGHT LINES AND THEIR ANGLES.

DEFINITIONS.

1. The divergence of two straight lines from a point constitutes an ANGLE. The quantity of the angle is the difference of direction between the two lines.

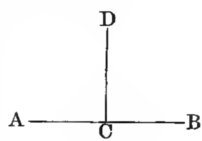
The lines themselves are said to contain the angle, and are called its SIDES. The point from which they diverge is called the VERTEX.



Thus, AB and BC are the sides, and B the vertex of the angle ABC.

In reading angles by three letters, the one at the vertex should always occupy the middle place.

2. When the adjacent angles made by one straight line with another are equal, they are called RIGHT ANGLES. Such are the angles ACD and BCD.



An angle which is greater than a right angle is termed OBTUSE; an angle less than a right angle, ACUTE.

3. One straight line is said to be PERPENDICULAR to another when it makes right angles with it; OBLIQUE, when it makes unequal angles.

4. Two straight lines are said to be PARALLEL to each other when they have the same direction. Thus, AB and CD are parallel.

Corollary. Parallel lines will never meet, however far produced; for, if they were to meet, they would make an angle with each other, that is, they would differ in direction, which is contrary to the definition.

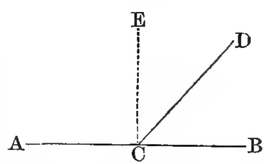
Scholium. Lines having the same direction may also be said to have exactly opposite directions when taken in different ways. Thus, the direction AB is opposite to the direction DC.

5. A straight line, or any other magnitude, is said to be BISECTED when it is divided into two equal parts.

THEOREM I.

The two angles which one straight line makes with another, on one side of it, are together equal to two right angles.

Let the straight line DC meet the straight line AB in C. Then will the two angles ACD, DCB, be together equal to two right angles.



For, if DC be perpendicular to AB, each of these angles will be a right angle (Definition 3, Section IV). But if DC is oblique to AB, erect the perpendicular CE. Now, the part ACE of the obtuse angle ACD, is a right angle; and the remaining part ECD and the

acute angle DCB together make up another right angle, ECB.

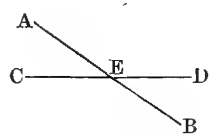
Therefore, the two angles which one straight line makes with another, on one side, are together equal to two right angles.

Cor. 1. In a similar manner, whatever be the number of adjacent angles formed at C, on one side of AB, it may be shown that their sum is two right angles.

Cor. 2. *A right angle is the fourth part of the whole compass in the plane about any point*, as C; for, half of this compass lies on one side of AB, and half on the other.

THEOREM II.

If two straight lines cut one another, the opposite or vertical angles are equal.



Let AB and CD be two straight lines cutting one another in E. Then will AEC be equal to its opposite or vertical angle DEB.

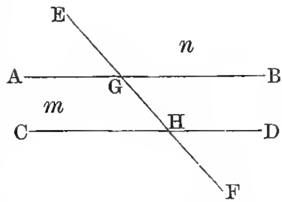
For the angles AEC, AED, on one side of CD, are together equal to two right angles (Theo. I); so, also, the angles AED, DEB, on one side of AB, are together equal to two right angles; therefore, the sum of AEC and AED is equal to the sum of AED and DEB (Axiom 3). Now take away the common angle AED, and the remaining angle AEC is equal to the remaining angle DEB (Ax. 5).

In the same manner it may be proved that the angle AED is equal to its opposite angle CEB.

Therefore, if two straight lines, etc.

THEOREM III.

If a straight line intersect two parallels, the corresponding inner and outer angles will be equal to each other; also the alternate angles.



Let the straight line EF intersect the two parallel straight lines AB, CD, in G and H. Then will any two corresponding inner and outer angles, as GHD and EGB, be equal.

For, since EF is a straight line, its part HG has the same direction from H that its part GE has from G (Def. 6, Sec. I); and since HD and GB are by hypothesis parallel, HD has the same direction from H that GB has from G (Def. 4, Sec. IV). Therefore, the difference in direction between HG and HD is equal to the difference in direction between GE and GB; that is, the angle GHD is equal to the angle EGB (Def. 1, Sec. IV).

Again, any two alternate angles, as AGH and GHD, are equal to each other. For AGH is equal to its opposite angle EGB (Theo. II). But EGB has just been proved equal to GHD. Consequently, AGH is equal to GHD (Ax. 3).

Therefore, if a straight line intersect, etc.

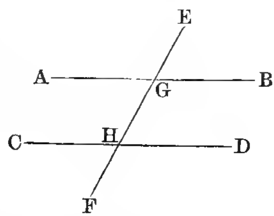
Cor. 1. It is evident that if AB is not parallel to CD, but takes some other direction, as mGn, the corresponding inner and outer angles will be unequal; also, the alternate angles.

Cor. 2. If two angles have their sides parallel,

each to each, and directed the same way from the vertex, they are equal.

THEOREM IV.

If a straight line intersect two parallels, the sum of the two inner angles on the same side will be equal to two right angles.



Let EF intersect the two parallels AB, CD, in G and H. Then will BGH and GHD be together equal to two right angles.

For the sum of the angles GHC and GHD is two right angles (Theo. I). But GHC is equal to its alternate angle BGH (Theo. III). Therefore, the sum of BGH and GHD is equal to two right angles.

Hence, if a straight line, etc.

Cor. 1. If BGH is a right angle, GHD must also be a right angle. Hence, *if a straight line is perpendicular to one of two parallels, it is perpendicular to the other.*

Cor. 2. If two straight lines are both perpendicular to a third, they are parallel.

EXERCISES.

1. Prove that the sum of all the adjacent angles made by any number of straight lines meeting in one point is equal to four right angles.

2. Prove that if one of the four angles made by

two straight lines intersecting each other be a right angle, each of the others will be a right angle.

3. Prove that the alternate outer angles EGA and FHD (Figure to Theo. III) are equal to each other.

4. Prove that the sum of the two outer angles on the same side, EGB and FHD (Fig. Theo. IV), is equal to two right angles.

5. Point out all the angles equal to EGB (Fig. Theo. IV); also, all the angles equal to EGA; and show in each case why they are equal.

SEC. V.—TRIANGLES.

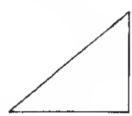
DEFINITIONS.

1. A PLANE FIGURE is a portion of a plane bounded on all sides by lines.

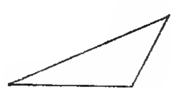
2. A POLYGON is a plane figure bounded by straight lines.

3. A TRIANGLE is a polygon of three sides.

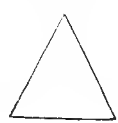
If it has one right angle it is called a *right-angled* triangle. The side opposite the right angle is called the HYPOTENUSE.



A triangle which has no right angle is called an *oblique-angled* triangle.



An *isosceles* triangle is one which has two equal sides.



4. Any side of a triangle may be considered as its *base*, and the opposite angle as its *vertex;* but in an isosceles triangle, that side is usually called the base which is not equal to either of the others.

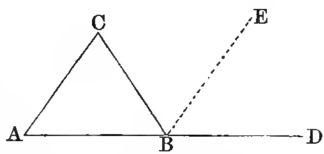
The ALTITUDE of a triangle is the perpendicular let fall from the vertex on the base, or the base produced.

5. A triangle, or other polygon, is called *equiangular* when all its angles are equal; *equilateral,* when all its sides are equal.

6. Two polygons are called *mutually equilateral* or *mutually equiangular,* if the sides or angles of the one are equal to the sides or angles of the other, each to each, taken in the same order.

THEOREM V.

The three interior angles of any triangle are together equal to two right angles.



Let ABC be any triangle. It is to be proved that the angles ABC, BCA, CAB, are together equal to two right angles.

Produce AB to any point D, and draw BE parallel to AC. Now, the sum of the three angles ABC, CBE, EBD, is equal to two right angles (Cor. 1, Theo. I). But because CB intersects the parallels AC and BE, the angle CBE is equal to its alternate angle BCA (Theo. III); also, because AD intersects the same parallels, the angle EBD is equal to its corresponding inner angle CAB. Therefore, the sum of ABC, BCA, CAB, is equal to two right angles.

That is, the three interior angles of a triangle, etc.

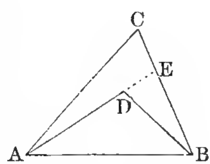
Cor. 1. *If one side of a triangle be produced, the exterior angle will be equal to the sum of the two opposite interior angles.* For the sum of CBE and EBD, that is, the whole exterior angle CBD, is equal to the sum of BCA and CAB.

Cor. 2. *A triangle can not have more than one right angle;* for, if it had two, the third angle would be nothing. Still less can a triangle have more than one obtuse angle.

THEOREM VI.

If two straight lines be drawn from the extremities of one side of a triangle to a point within, their sum will be less than that of the other two sides of the triangle.

From the extremities of AB, let straight lines be drawn to a point D within the triangle ABC. It is to be proved that the sum of AD and DB is less than the sum of AC and CB.



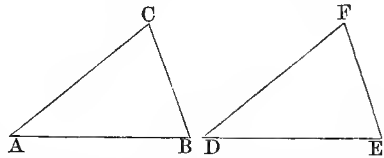
Produce AD to meet BC in E. Now, since AE is a straight line, it is less than the sum of AC and CE, which form a broken line (Ax. 10). Therefore, if we add EB to both, it is evident that the sum of AE and EB is less than the sum of AC, CE, and EB, that is, less than the sum of AC and CB (Ax. 6). In the same manner it may be shown that the sum of AD and DB is less than the sum of AE and EB; still more, then, we may conclude it is less than the sum of AC and CB.

Therefore, if two straight lines, etc.

THEOREM VII.

If two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, they are equal throughout.

Let the triangles ABC, DEF, have the side AB equal to the side DE, and AC equal to DF, and the included angle A to the included angle D.



Then will the two triangles be equal throughout; that is, the third side BC will be equal to the third side EF, the angle B to the angle E, the angle C to the angle F, and the triangle ABC as a whole to the triangle DEF as a whole.

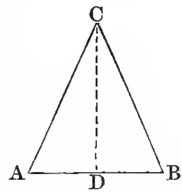
If the triangle ABC be applied to the triangle DEF so that the point B shall be upon the point E, and the side BA upon the side ED, then since these sides are by hypothesis equal, the point A will fall upon the point D; and since the angle A is equal to the angle D, the side AC will fall upon the side DF; and since these sides again are equal, the point C will fall upon the point F. But the point B is also upon the point E. Therefore, the side CB must coincide with the side FE (Ax. 10), and the angle C with the angle F, and the angle B with the angle E, and the triangle ABC as a whole with the triangle DEF as a whole; and since they coincide, each with each, they are also equal (Ax. 11).

Therefore, if two triangles, etc.

Cor. In equal triangles, equal angles are opposite to equal sides.

THEOREM VIII.

The angles at the base of an isosceles triangle are equal to each other.



Let ABC be an isosceles triangle having the side AC equal to the side BC. It is to be proved that the angle A is equal to the angle B.

Draw CD bisecting the angle ACB. Then in the triangles ADC, BDC, since the side AC is, by hypothesis, equal to the side BC, and the side CD is common to both, and the included angle ACD is, by construction, equal to the included angle BCD, it follows that the triangle ADC is equal to the triangle BDC (Theo. VII), and the angle A to the angle B (Cor. Theo. VII).

Therefore, the angles at the base, etc.

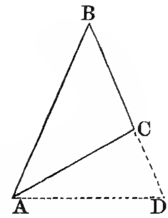
Cor. 1. *A straight line bisecting the vertical angle of an isosceles triangle is a perpendicular to the middle point of the base.* For AD is equal to DB; also, the angles CDA, CDB, being equal, are right angles (Def. 2, Sec. 4).

Cor. 2. An equilateral triangle is also equiangular.

THEOREM IX.

The greater of two unequal sides in a triangle has the greater angle opposite to it.

Let ABC be a triangle having the side AB greater than the side BC. Then will the angle BCA be greater than the angle BAC.



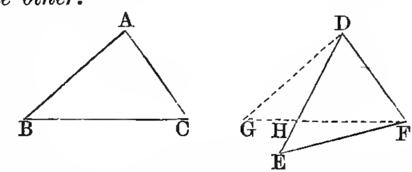
Produce BC so as to make BD equal to BA; also, join DA. Now, in the isosceles triangle ABD, the angles BDA, BAD, are equal (Theo. VIII). But the exterior angle BCA, of the triangle ACD, is greater than the opposite interior angle BDA (Cor. I, Theo. V), and consequently greater than BAD; still more, then, is it greater than BAC.

Therefore, the greater of two unequal sides, etc.

Cor. Hence, if two angles of a triangle are equal, the sides opposite to them can not be unequal.

THEOREM X.

If two triangles have two sides of the one equal to two sides of the other, but the included angles unequal, the base of that which has the greater angle is greater than the base of the other.



Let the two triangles ABC, DEF, have the two sides AB, AC, respectively equal to the two sides DE, DF, but the angle BAC greater than the angle EDF. It is to be proved that the base BC is greater than the base EF.

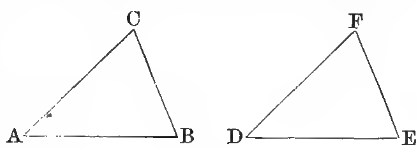
Because the angle EDF is, by hypothesis, less than the angle BAC, the sum of the angles E and F must be greater than the sum of the angles B and C (Theo.

V): hence, either E is greater than B, or F is greater than C. Suppose, then, that F is greater than C. On the adjacent side DF, describe the triangle DGF equal to the triangle ABC, and so placed that DG shall be the side equal to AB. Now, the sum of DH and HG is greater than DG (Ax. 10); and the sum of EH and HF is greater than EF. Therefore, the sum of the entire lines DE and GF is greater than the sum of DG and EF. Taking away from these unequals the equals DE and DG, we have the remainder GF greater than the remainder EF. But GF is equal to BC by construction. Therefore, BC is greater than EF.

Hence, if two triangles, etc.

THEOREM XI.

If two triangles have two angles and the included side of the one equal to two angles and the included side of the other, each to each, they are equal throughout.



Let ABC, DEF, be two triangles having the angles A and B respectively equal to the angles D and E, and the included side AB equal to the included side DE. It is to be proved that these triangles are equal throughout.

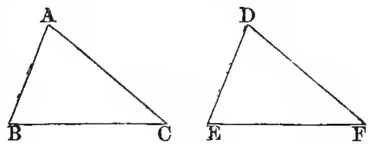
If the triangle ABC be applied to the triangle DEF, so that the side AB shall coincide with its equal DE, then because the angle A is equal to the angle D, the side AC will fall on the side DF, and the point C will be found somewhere in the line DF; also, because the

angle B is equal to the angle E, the side BC will fall on the side EF, and the point C will be found somewhere in the line EF; consequently the point C, being in both the lines DF and EF, must be at their point of intersection F. Hence, the two triangles exactly coincide, and are equal throughout (Ax. 11); that is, the side AC is equal to the side DF, the side BC to the side EF, the angle C to the angle F, and the triangle ABC as a whole to the triangle DEF as a whole.

Therefore, if two triangles, etc.

THEOREM XII.

Two triangles which are mutually equilateral are also mutually equiangular.



Let the two triangles ABC, DEF, have the three sides AB, BC, CA, respectively equal to the three sides DE, EF, FD. Then, also, will the angles A, B, and C be respectively equal to the angles D, E, and F.

If the angle A is not equal to the angle D, it must be either greater or less. But it can not be greater than D, for then the base BC would be greater than the base EF (Theo. X), which is contrary to the hypothesis. Neither can it be less than D; for, in that case, BC would be less than EF, which is also contrary to the hypothesis. Hence, A must be equal to D; and, in the same manner, it may be proved that B is equal to E, and C to F.

Therefore, two triangles, etc.

Cor. If two triangles are mutually equilateral they are equal throughout.

EXERCISES.

1. Prove that if two angles of one triangle are equal to two angles of another, the third angles are equal.

2. Prove that each angle of an equilateral triangle is equal to two-thirds of a right angle.

3. Prove that if a perpendicular be erected on the middle point of a straight line, any point in it will be equally distant from the extremities of that line (Theo. VII).

4. Prove that from a given point without a straight line, only one perpendicular to that line can be drawn (Cor. 2, Theo. V).

5. If the vertical angle of an isosceles triangle be two-sevenths of a right angle, what will be the value of each of the angles at the base?

SEC. VI.—QUADRILATERALS.

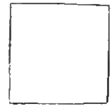
DEFINITIONS.

1. A QUADRILATERAL is a polygon of four sides.

2. A PARALLELOGRAM is a quadrilateral having its opposite sides parallel.

A parallelogram whose angles are all right angles is called a *rectangular* parallelogram, or simply a RECTANGLE.

A SQUARE is a rectangle whose sides are all equal.



3. A TRAPEZOID is a quadrilateral having only two of its opposite sides parallel.

4. A DIAGONAL of a quadrilateral or other polygon is a straight line joining two angles not adjacent to each other.

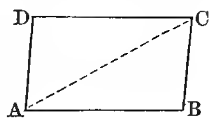
5. The AREA of any plane figure is the amount of surface which it contains.

6. Two plane figures are said to be *equivalent* when their areas are equal.

THEOREM XIII.

The opposite sides and angles of a parallelogram are equal to each other.

Let ABCD be a parallelogram. It is to be proved that any one of its sides is equal to the side opposite, and any one of its angles to the angle opposite.



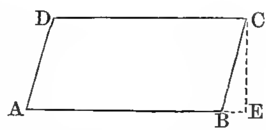
Draw the diagonal AC. Now, since AC intersects the two parallels AD, BC, the alternate angles DAC, BCA, are equal (Theo. III); also, since it intersects the parallels AB, DC, the alternate angles BAC, DCA, are equal. Hence, the two triangles ABC, ADC, have two angles of the one equal to two angles of the other, and the included side AC common; they are,

therefore, equal (Theo. XI); and the side AB is equal to the side DC (Cor., Theo. VII), and the side AD to the side BC, and the angle B to the angle D; also, since the angle DAC is equal to the angle BCA, and the angle BAC to the angle DCA, it follows that the whole angle BAD is equal to the whole angle BCD.

Therefore, the opposite sides and angles, etc.

Cor. 1. A diagonal of a parallelogram divides it into two equal triangles.

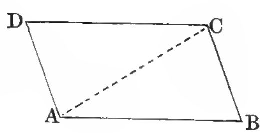
Cor. 2. If ABCD be a rectangular parallelogram, AD and BC will both be perpendicular to each of the parallels AB, DC (Def. 2, Sec. VI). Hence, *the perpendicular distance between two parallels is everywhere the same.*



Schol. Any side, as AB, of a parallelogram may be taken as its *base;* and a perpendicular, as CE, let fall from any point in the opposite parallel on the base, or the base produced, is called the altitude of the parallelogram.

THEOREM XIV.

If a quadrilateral has two of its sides equal and parallel, it is a parallelogram.



Let the quadrilateral ABCD have the sides AB, DC, equal and parallel. Then will it be a parallelogram.

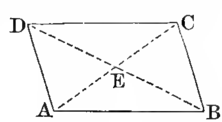
Join AC; then, since AC intersects the two parallels AB, DC, the alternate angles BAC, DCA, are equal (Theo. III). Now, in the triangles ABC, ADC, we have the side AB equal to the side DC, and the side AC common, and the included angle BAC equal to the included angle DCA; therefore, the two triangles are equal (Theo. VII), and the angle BCA is equal to the angle CAD (Cor., Theo. VII), and these being alternate angles, it follows that AD and BC are parallel (Cor. 1, Theo. III). Hence, ABCD has its opposite sides parallel, and is a parallelogram (Def. 2, Sec. VI).

Therefore, if a quadrilateral, etc.

Schol. If a quadrilateral has its opposite sides equal it is a parallelogram. For the triangles ABC, ADC, will in that case be mutually equilateral, and therefore equal (Cor., Theo. XII); and the parallelism of the opposite sides may be shown by the equality of the alternate angles, as above.

THEOREM XV.

The diagonals of a parallelogram bisect each other.



Let ABCD be a parallelogram. It is to be proved that its diagonals, AC, BD, bisect each other in E (Def. 6, Sec. IV).

In the triangles AEB, DEC, since the angle ABD is equal to its alternate angle BDC (Theo. III), and the angle BAC equal to its alternate angle ACD, and the included side AB equal to the included side DC

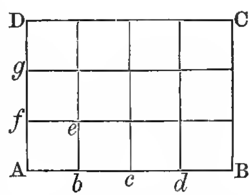
(Theo. XIII), it follows that the two triangles are equal (Theo. XI); consequently, the side AE is equal to the side EC, and BE to ED (Cor., Theo. VII); that is, the two diagonals bisect each other in E.

Therefore, the diagonals, etc.

THEOREM XVI.

The area of a rectangle is equal to the product of its base by its altitude.

Let ABCD be a rectangle. It is to be proved that its area is equal to the product of its base AB by its altitude AD.



Let AB be divided into a certain number of equal parts A*b*, *bc*, etc., taken as the units of length; also, let AD be divided into a certain number of the same units, A*f*, *fg*, etc. From *b*, *c*, etc., draw straight lines parallel to AD, and from *f*, etc., draw straight lines parallel to AB. Now, it is evident that the whole rectangle is divided into small squares (Cor. 2, Theo. XIII), each equal to A*bef*; which may be taken as the unit of area. Of these equal squares there are as many in the tier next to AB as there are units of length in AB; and there are as many equal tiers in the whole figure as there are units of length in AD. Therefore, the whole number of square units in ABCD is equal to the number of linear units in AB multiplied by the number of linear units in AD.

Hence, the area of a rectangle, etc.

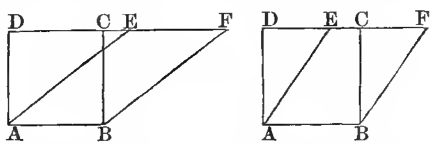
Schol. If AB and AD are incommensurable, that is, if no unit can be found into which they can both be divided without leaving a remainder in one of them, the theorem will still hold true; for if the unit be taken smaller and smaller, the remainder can be made less than any assignable quantity.

When the linear unit is one inch, the unit of area is a square inch; when the linear unit is one foot, the unit of area is a square foot, etc.

Cor. Since the base and altitude of a square are equal (Def. 2, Sec. VI), its area may be found by multiplying one side into itself.

THEOREM XVII.

The area of any parallelogram is equal to the area of a rectangle having the same base and altitude.



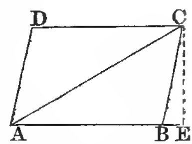
Let ABCD be a rectangle, and ABFE any parallelogram, on the same base AB, and of the same altitude, namely the perpendicular distance between the parallels AB, DF. Then will ABFE be equivalent to ABCD.

Since AB and DC are opposite sides of a parallelogram they are equal (Theo. XIII); and for the same reason AB and EF are equal; therefore DC is equal to EF (Ax. 3). Taking away each of these in turn from the whole line DF, we have the remainder DE equal to the remainder CF. But DA is equal to CB (Theo. XIII), and the included angle ADE is equal to

the included angle BCF (Theo. III); therefore, the triangles ADE, BCF, are equal (Theo. VII); and hence if each of them be taken away in turn from the whole figure ABFD, the remainder ABFE will be equivalent to the remainder ABCD.

Therefore, the area of any parallelogram, etc.

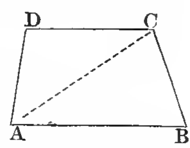
Cor. 1. The area of any parallelogram is equal to the product of its base by its altitude.



Cor. 2. Since any triangle, as ABC, is half of a parallelogram ABCD (Cor. 1, Theo. XIII) on the same base AB and having the same altitude CE (Def. 4, Sec. V), it follows that *the area of a triangle is equal to half the product of its base by its altitude.*

THEOREM XVIII.

The area of a trapezoid is equal to half the product of the sum of its parallel sides by its altitude.



Let ABCD be a trapezoid of which AB and DC are the parallel sides. Then will its area be equal to half the product of the sum of AB and DC into its altitude, namely, the perpendicular distance between AB and DC.

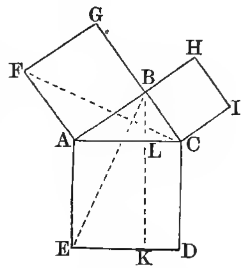
Draw the diagonal AC. Now, the area of the triangle ABC (Cor. 2, Theo. XVII) is equal to half the product of its base AB into its altitude, which is the same as the altitude of the trapezoid; again, the area of the triangle ADC is equal to half the product of its base DC into its altitude, which is also the same

as the altitude of the trapezoid. Therefore, adding together the area of the two triangles, we have the area of the whole figure equal to half the product of the sum of AB and DC into the altitude.

Hence, the area of a trapezoid, etc.

THEOREM XIX.

The square described on the hypotenuse of a right-angled triangle is equivalent to the sum of the squares described on the other two sides.



Let ABC be a triangle right-angled at B. It is to be proved that the square AEDC is equivalent to the sum of the squares ABGF and BHIC.

Join FC, BE, and draw BK parallel to AE. Also observe that since ABC and ABG are right angles, BC and BG form one straight line.

Now, in the triangles EAB, CAF, the side EA is equal to the side CA, since they are sides of the same square, and for the same reason the side AB is equal to the side AF. But the included angles EAB, CAF, are also equal; for each of them is composed of a right angle and the angle CAB. Therefore, the two triangles are equal (Theo. VII).

But since the triangle CAF and the square BAFG have the same base AF, and the same altitude, namely the perpendicular distance between the parallels AF,

CG, the square must be double of the triangle (Cors. 1 and 2, Theo. XVII). For like reason the parallelogram AEKL is double of the triangle EAB. But doubles of equals are equals; therefore, the parallelogram AEKL is equivalent to the square BAFG. In the same manner (by joining AI and BD), it may be shown that the parallelogram CDKL is equivalent to the square BCIH. Hence, the whole square ACDE is equivalent to the sum of the squares BAFG, BCIH.

Therefore, the square described, etc.

Cor. 1. The hypotenuse is equal to the square root of the sum of the squares of the other two sides.

Cor. 2. The square of either of the sides containing the right angle is equivalent to the square of the hypotenuse diminished by the square of the other side. By taking the square root of the remainder the side itself will be found.

Cor. 3. If two right-angled triangles have the hypotenuse and one side of the one respectively equal to the hypotenuse and one side of the other, the third sides will also be equal.

EXERCISES.

1. If the side of a square be 36 inches, what is its area in square inches? What in square feet?

2. If the base of a parallelogram be 3 feet and its altitude 4 feet and 6 inches, what is its area?

3. If the base of a triangle be 50 yards and its altitude 20 yards, what is its area?

4. If the parallel sides of a trapezoid be 12 rods

and 16 rods, and its altitude $8\frac{1}{2}$ rods, what is its area?

5. If the sides containing the right angle of a right-angled triangle be 3 and 4, what is the length of the hypotenuse?

6. If the hypotenuse be 10 and one of the sides 8, what is the length of the other side?

7. Prove that if a parallelogram has one right angle it is a rectangle.

8. Prove that the diagonals of a rectangle are equal to each other.

9. Prove that a perpendicular is the shortest line that can be drawn to a straight line from a point without it; also, that of two oblique lines that which is furthest from the perpendicular is the longest (Theo. XIX).

SEC. VII.—OF POLYGONS IN GENERAL.

DEFINITIONS.

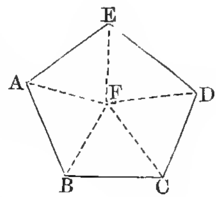
1. A polygon of five sides is called a PENTAGON; one of six sides, a HEXAGON; of seven sides, a HEPTAGON; of eight sides, an OCTAGON; of ten sides, a DECAGON; of twelve sides, a DODECAGON.

2. A *regular* polygon is one which is both equilateral and equiangular. Squares and equilateral triangles are species of regular polygons.

3. The PERIMETER of a polygon is the sum of all its sides.

THEOREM XX.

The sum of all the angles of any polygon is equal to twice as many right angles, wanting four, as the figure has sides.



Let ABCDE be any polygon. It is to be proved that the sum of all its angles A, B, C, D, E, is equal to twice as many right angles, wanting four, as it has sides.

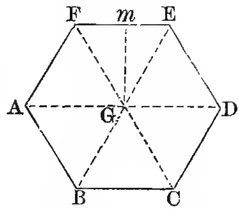
From the vertex of each angle draw straight lines AF, BF, etc., to any point F within, thus dividing the polygon into as many triangles as it has sides. Now, since the angles of each triangle are together equal to two right angles (Theo. V), it follows that the angles of all the triangles are together equal to twice as many right angles as the figure has sides. But the sum of the angles A, B, C, D, E, is equal to the sum of the angles of all the triangles, wanting the angles about the point F; and the angles about the point F are together equal to four right angles (Cor. 2, Theo. I). Therefore, the sum of the angles A, B, C, D, E, is equal to twice as many right angles, wanting four, as the polygon has sides.

That is, the sum of all the angles of any polygon, etc.

Schol. The area of a polygon may be found by dividing it into triangles as above (or by diagonals drawn from any angle A to the opposite angles), then finding the areas of these triangles separately and adding them together.

THEOREM XXI.

A regular polygon may be divided into as many equal triangles as it has sides.



Let ABCDEF be a regular polygon. Draw AG and BG bisecting the angles A and B; also, draw GC, GD, etc., to the other angles.

Now, since the triangles AGB, BGC, have the side AB equal to the side BC (Def. 2, Sec. VII), and the side BG common, and the included angle ABG equal to the included angle CBG, the two triangles are equal to each other (Theo. VII). Hence the angle GAB is equal to the angle GCB (Cor., Theo. VII). But GAB is by construction half of BAF; therefore, GCB must be half of the equal angle BCD, that is, BCD is bisected by GC. Hence, it may be shown in the same manner as before that the triangles BGC, CGD, are equal: and so on of the other triangles in succession; of which there is one on each side of the polygon.

Therefore, a regular polygon, etc.

Schol. The *center* of a regular polygon is the point G, found by bisecting any two of its angles. A perpendicular, as G*m*, let fall from the center upon any side of a regular polygon is called its APOTHEGM. It is the altitude of the triangle EFG, and consequently of each of the equal triangles into which the polygon is divided.

Cor. Since the area of each triangle is equal to

half the product of its base by its altitude, the area of all the triangles taken together is equal to half the product of the sum of their bases by their common altitude: that is, *the area of a regular polygon is equal to half the product of its perimeter by its apothegm.*

EXERCISES.

1. To how many right angles are all the angles of a quadrilateral equal? Of a pentagon? Of a hexagon? Of a heptagon? etc.

2. What is the area of a regular pentagon whose side is 25, and the apothegm 17.205?

3. Prove that each angle of a regular hexagon is equal to four thirds of a right angle.

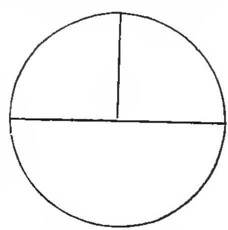
SEC. VIII.—OF THE CIRCLE.

DEFINITIONS.

1. A CIRCLE is a plane figure bounded by a curve, called the CIRCUMFERENCE, every point in which is equally distant from a point within, called the CENTER.

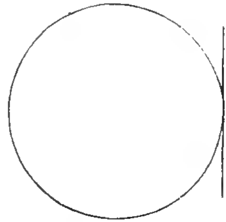
2. A straight line drawn from the center to any point of the circumference is called the RADIUS.

A double radius, that is, a straight line passing through the center, and terminated both ways by the circumference, is called the DIAMETER.



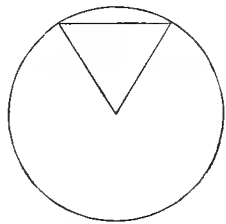
Cor. In the same circle, all radii are equal; also, all diameters.

3. A TANGENT is a straight line which meets the circumference, but being produced does not cut it.



4. Any portion of the circumference is called an ARC. The CHORD of an arc is the straight line joining its extremities. A chord produced one way beyond the circumference is called a SECANT.

5. A SEGMENT of a circle is the part contained by an arc and its chord.



A SECTOR is the part contained by an arc and two radii drawn to its extremities.

6. Half a circle is called a SEMICIRCLE; half a circumference, a SEMI-CIRCUMFERENCE. A quarter of a circle or of a circumference is called a QUADRANT.

7. An INSCRIBED ANGLE is one which has its vertex in the circumference, and is contained by two chords.

A POLYGON is said to be *inscribed* in a circle when all its sides are chords of that circle.

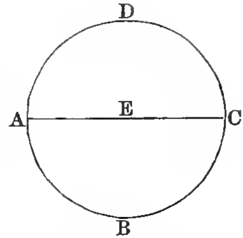
A CIRCLE is said to be *inscribed* in a polygon when all the sides of the polygon are tangents to the circle.

When one figure is inscribed in another, the latter is said to be *circumscribed* about the former.

THEOREM XXII.

A diameter divides the circle and its circumference into two equal parts.

Let ABCD be a circle of which E is the center and AC a diameter. It may be shown that the segment ABC is equal to the segment ADC, and the arc ABC to the arc ADC.



Let the segment ABC be applied to the segment ADC, the line AC remaining common; then the arc ABC will exactly coincide with the arc ADC: for if not, suppose some point in ABC to fall within or without ADC; then we shall have points in the circumference unequally distant from the center, which is contrary to the definition of a circle. Hence, the arc ABC is equal to the arc ADC, and the segment ABC to the segment ADC (Ax. 11).

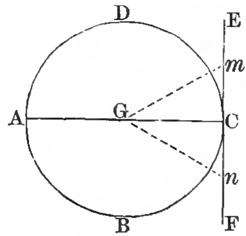
Therefore, a diameter divides, etc.

Schol. In a similar manner it may be proved that two circles of equal diameters or radii are equal to each other.

THEOREM XXIII.

If a straight line is perpendicular to a diameter at its extremity, it is a tangent to the circle.

Let the straight line EF be perpendicular to the diameter AC at its extremity C. Then will EF be a tangent to the circle ABCD.



In EF take any two points m and n, one on each side of C, and connect them by

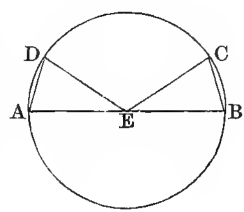
straight lines with the center G. Now, since G*C*m is a right-angled triangle, the square of Gm is greater than the square of GC (Theo. XIX); hence, Gm is greater than GC, that is, greater than the radius. Therefore, the point m, however near it may be to C, is necessarily without the circle. The same may be proved, in like manner, of the point n, or any other point in EF on either side of C. Hence, EF can meet the circumference only in the point C, and it is consequently a tangent (Def. 3, Sec. VIII).

Therefore, if a straight line, etc.

Cor. If a chord be drawn perpendicular to a tangent at the point of contact, it will be a diameter.

THEOREM XXIV.

Equal arcs in a circle have equal chords, and subtend equal angles at the center.



Let AD and BC be equal arcs, and E the center of the circle. It is to be proved that the chords AD, BC are equal; also, the subtended angles AED, BEC.

Let the sector EBC be applied to the sector EAD, so that the line EB shall fall upon the line EA, the point E remaining common; then, since these lines are equal, being radii (Cor., Def. 2, Sec. 8), the point B will fall upon the point A; and, since all points in the circumference are equally distant from the center, the arc BC will fall

upon the arc AD; and these arcs being by hypothesis equal, the point C will fall upon the point D; therefore (Ax. 10), the chord BC will coincide with the chord AD, and be equal to it.

Again, in the triangles EAD, EBC, the two sides EA, ED, are equal to the two sides EB, EC, since they are all radii; and the base AD has just been proved equal to the base BC; therefore, the two triangles are mutually equiangular (Theo. XII), and the angle AED is equal to the angle BEC (Cor., Theo. VII).

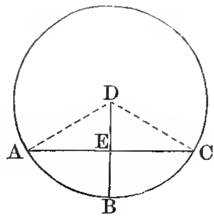
Hence, equal arcs in a circle, etc.

Schol. If the whole circumference be divided into any number of equal arcs, and radii be drawn to all the points of division, the whole compass about the center will be divided into the same number of equal angles. Hence, *an arc may be taken as the measure of its subtended angle.* To this end the circumference is conceived to be divided into 360 equal parts, called *degrees*, each degree into 60 *minutes*, and each minute into 60 *seconds.* Then whatever number of degrees, etc., an arc contains, the same number will denote the magnitude of the angle at the center which it subtends.

Cor. A quadrant is an arc of 90 degrees; and a right angle, being measured by a quadrant, is an angle of 90 degrees.

THEOREM XXV.

If a radius is perpendicular to a chord, it bisects the chord, and also the arc which the chord subtends.



Let the radius DB be perpendicular to the chord AC. It is to be proved that it bisects AC, and also the arc ABC.

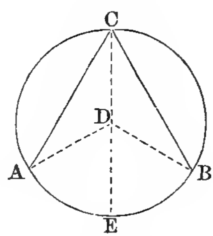
Draw the radii DA, DC. Now, since the right-angled triangles AED, CED, have the hypotenuse AD equal to the hypotenuse CD, and the side ED common, it follows that the third side AE is equal to the third side CE (Cor. 3, Theo. XIX); that is, AC is bisected in E.

Again, since the triangles AED, CED, are mutually equilateral, the angle ADE is equal to the angle CDE (Theo. XII). Hence, the arc AB which measures the former angle is equal to the arc CB which measures the latter (Schol., Theo. XXIV).

Therefore, if a radius is perpendicular, etc.

THEOREM XXVI.

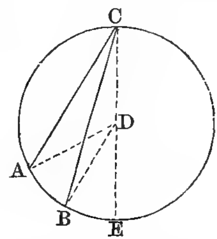
An inscribed angle is measured by half the arc on which it stands.



Let ACB be an angle inscribed in a circle. It is measured by half the arc AEB.

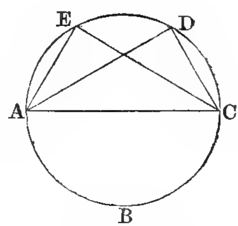
First, suppose the center D to be within the angle. Draw the diameter CE; also, join DA and DB. Now, since AD and DC are radii, the triangle ADC is isosceles; hence, the angles DAC, DCA, are equal (Theo. VIII). But the exterior angle ADE is equal to the sum of the two opposite interior angles DAC, DCA (Cor. 1,

Theo. V); it is consequently double of DCA; and since ADE is measured by the arc AE (Schol., Theo. XXIV), it follows that DCA is measured by half of AE. In the same manner it may be shown that DCB is measured by half of BE. Therefore, the whole angle ACB is measured by half of the whole arc AEB.



Next, let the center D be without the angle ACB. By the above demonstration, the angle ACE is measured by half the arc AE, and the angle BCE is measured by half the arc BE; therefore, BCA, which is the difference of these two angles, is measured by half the difference of the two arcs, that is, by half of AB.

Hence, an inscribed angle, etc.

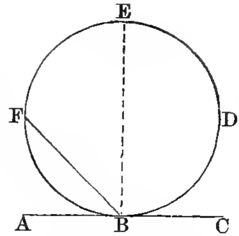


Cor. 1. *Angles inscribed in the same segment*, as AEC and ADC, *are equal;* for they are measured by half the same arc ABC.

Cor. 2. *Any angle inscribed in a semicircle is a right angle;* being measured by half a semi-circumference.

THEOREM XXVII.

An angle contained by a tangent and a chord is measured by half the intercepted arc.



Let AC be a tangent to a circle, and BF a chord drawn from the point of contact B. Then will the angle ABF be measured by half the arc BF, and the angle CBF by half the arc BDF.

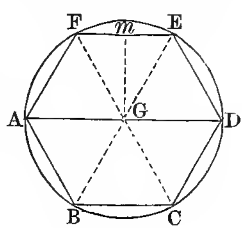
Draw BE perpendicular to AC, and it will be a diameter of the circle (Cor., Theo. XXIII). Now, since ABE is a right angle, it is measured by half the semi-circumference, BFE (Cor., Theo. XXIV); and since the part EBF is an inscribed angle, it is measured by half the arc FE (Theo. XXVI); therefore, the remaining angle ABF is measured by half the remaining arc BF.

Again, because CBE is measured by half the semi-circumference BDE, and EBF by half the arc EF, it follows that the whole angle CBF is measured by half the whole arc BDF.

Therefore an angle contained, etc.

THEOREM XXVIII.

If the circumference be divided into any number of equal arcs, their chords will form a regular polygon.



Let the arcs AB, BC, etc., be all equal to each other. Then will the inscribed polygon formed by their chords AB, BC, etc., be a regular polygon.

Draw the radii AG, BG, CG, etc. Now, since two sides in

each of the triangles AGB, BGC, etc., are radii; and since their bases AB, BC, etc., are equal, being chords of equal arcs (Theo. XXIV); it follows that these triangles are mutually equiangular (Theo. XII). But they are also isosceles; hence (Theo. VIII), all the angles at their bases, GBA, GBC, GCB, GCD, etc., are equal; hence, also, the sums, ABC, BCD, etc., that is, all the angles of the polygon, are equal. And its sides have already been shown to be equal. It is, therefore, a regular polygon (Def. 2, Sec. VII).

Hence, if the circumference, etc.

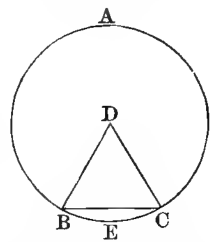
Cor. 1. If the polygon ABCDEF be a hexagon, the angle AGB, being measured by one-third of the semi-circumference, will be one-third of two right angles (Cor., Theo. XXIV). Hence, either of the two equal angles at the base of the triangle, as GBA, will also be one-third of two right angles (Theo. V). Therefore, AB is equal to AG (Cor., Theo. IX); that is, *the side of a regular hexagon is equal to the radius of the circumscribed circle.*

Cor. 2. A perpendicular, as G*m*, let fall from the center of the circumscribed circle on any side of a regular polygon, will be the apothegm of the polygon (Schol., Theo. XXI).

Cor. 3. If from the center G, with the apothegm G*m* as radius, a circle be drawn, the side FE, and consequently all the other sides of the regular polygon, will be tangents to that circle (Theo. XXIII), and the circle will be inscribed in the polygon (Def. 7, Sec. VIII).

THEOREM XXIX.

The area of a circle is equal to half the product of its radius by its circumference.



Let ABC be a circle, of which DB or DC is radius.

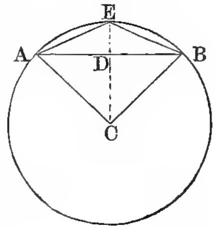
If we conceive the whole circumference to be divided into equal arcs so small that the points of division may be regarded as infinitely near to each other, the perimeter of the regular polygon formed by their chords (Theo. XXVIII) will coincide with the circumference, and the area of the polygon will be equal to the area of the circle; also, the apothegm of the polygon (Cor. 2, Theo. XXVIII) will be equal to the radius of the circle. But the area of the polygon will be equal to half the product of its apothegm by its perimeter (Cor., Theo. XXI). Therefore, the area of the circle must be equal to half the product of its radius by its circumference.

Cor. 1. In like manner it may be shown that the area of any sector, as BECD, is equal to half the product of the radius into its arc.

Cor. 2. The area of a segment BEC, less than a semicircle, may be found by subtracting the triangle BDC from the sector BECD. The area of a segment BAC, greater than a semicircle, may be found by adding the triangle BDC to the sector BACD.

THEOREM XXX.

The radius, and one side of a regular inscribed polygon being given, the side of a regular inscribed polygon of twice the number of sides may be found.



Let AB be the side of a regular inscribed polygon in a circle whose radius is CA or CB. Draw CE perpendicular to AB. This will bisect AB and also the arc AEB (Theo. XXV). Hence, the chords AE, EB (Theo. XXVIII), will be sides of a regular inscribed polygon having twice as many sides as the first.

Now, if the lengths of CB and AB are given, we shall have, in the right-angled triangle CDB, the hypotenuse CB and the side BD; hence, we may find the other side DC (Cor. 2, Theo. XIX). Subtracting this from the radius CE, we shall have DE. Then, in the right-angled triangle BDE, we shall have the two sides BD, DE, from which we can find the hypotenuse EB (Cor. 1, Theo. XIX); which is the side required.

Schol. If the diameter be 1, the side of a regular inscribed hexagon will be $\frac{1}{2}$ (Cor. 1, Theo. XXVIII). From this we can find the side, and consequently the perimeter, of a regular inscribed dodecagon; from that we can find the perimeter of a regular inscribed polygon of 24 sides, etc. By carrying this calculation on, to polygons of an indefinitely large number of sides, it is found that the perimeter, though it increases at every step, never exceeds 3.14159, except by decimal figures beyond the 9; that is, beyond the fifth place of decimals. Hence, since the perimeter ultimately

coincides with the circumference, it follows that the circumference can not differ from 3.14159, except by decimals beyond the fifth place. Disregarding these, as of trifling value, we may conclude that *the circumference of a circle whose diameter is* 1, *is* 3.14159.

Cor. If the diameter is 1, the area is equal to $\frac{\frac{1}{2}\times 3.14159}{2}$ (Theo. XXIX), which by reduction is .7854.

EXERCISES.

1. How many degrees are contained in an arc cut off by the side of an inscribed square? Of a regular inscribed pentagon? Hexagon? Heptagon? etc.

2. What is the area of a sector whose arc is 25 feet, in a circle whose diameter is 16 feet?

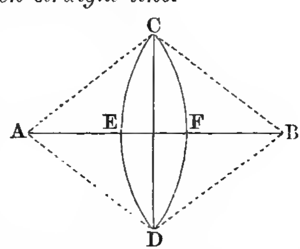
3. Prove that in an inscribed quadrilateral the sum of any two opposite angles is 180 degrees (Theo. XXVI).

4. Prove that of two unequal chords the less is the further from the center (Theo. XXV, and Cor. 2, Theo. XIX).

SEC. IX.—PROBLEMS IN CONSTRUCTION.

PROBLEM I.

To bisect a given straight line.

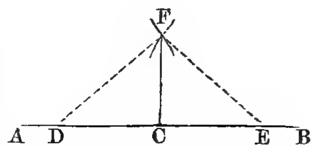


Solution. Let AB be the given line. From A as a center, with a radius obviously greater than one-half of AB, describe the arc CFD. From B as a center, with the same radius, describe the arc CED. Join the points of intersection C and D.

Now, since the opposite sides of the quadrilateral ACBD are equal, it is a parallelogram (Schol., Theo. XIV). But the diagonals of a parallelogram bisect each other (Theo. XV). Therefore, CD bisects AB.

PROBLEM II.

At a given point in a straight line, to erect a perpendicular to that line.

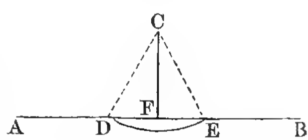


Solution. Let AB be an indefinite straight line, and C the given point in it. Take the points D and E equally distant from C. From D as a center, with a radius greater than DC, describe an arc; and from E as a center, with the same radius, describe another arc intersecting the former at some point F. Now, draw FC, and it will be the perpendicular required.

Join FD and FE. Then, since the triangles DFC, EFC, are by construction mutually equilateral, the angles FCD, FCE, are equal (Theo. XII); they are, therefore, right angles, and CF is perpendicular to AB at the point C (Defs. 2 and 3, Sec. IV.)

PROBLEM III.

To draw a perpendicular to a straight line from a given point without.



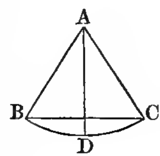
Solution. Let C be the given point without the straight line, AB. From C as a center, with a radius greater than the

shortest distance to the line, draw an arc cutting AB in D and E. Bisect DE in F (Prob. I). Then draw CF, and it will be the perpendicular required.

Join CD and CE. Now, since the triangles DCF, ECF, are mutually equilateral, the angles CFD, CFE, are equal (Theo. XII); they are, therefore, right angles, and CF is a perpendicular to AB from the point C.

PROBLEM IV.

To bisect a given angle.

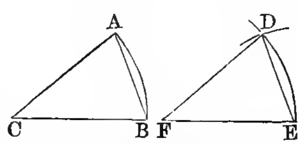


Solution. Let BAC be the given angle. From A as a center, with any radius, draw an arc BDC intersecting both sides of the angle. Then draw AD perpendicular to the chord BC (Prob. III), and it will bisect the angle.

For, since the perpendicular AD is radius, it bisects the arc BDC (Theo. XXV). Therefore, the arcs BD, DC, being equal, the subtended angles BAD, DAC, must also be equal (Theo. XXIV), and the angle BAC is bisected by AD.

PROBLEM V.

At a given point in a straight line, to make an angle equal to a given angle.

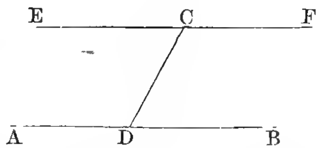


Solution. Let ACB be the given angle, and F the given point in a straight line FE. From C, as a center, with any radius as CB, describe an arc BA intersecting both the sides of the angle; also, draw the chord BA. Then, from F as a center, with

a radius FE equal to CB, draw an indefinite arc; and from the center E, with a radius equal to the chord BA, draw an arc intersecting the other at D; also, join DE and DF. The angle DFE is the angle required. For, by equality of the triangles DEF, ABC, the angle F is equal to the angle C (Theo. XII).

PROBLEM VI.

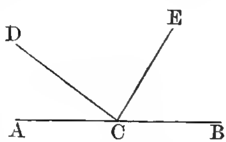
Through a given point to draw a straight line parallel to a given line.



Solution. Let AB be the given straight line, and C the given point. In AB take any point D, and join CD. Through C draw EF, making the angle ECD equal to the angle CDB (Prob. V). These being alternate angles, the straight line EF must be parallel to AB (Cor. 1, Theo. III).

PROBLEM VII.

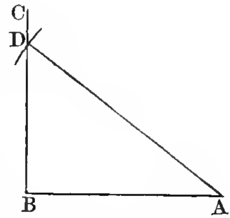
Two angles of a triangle being given, to find the third.



Solution. At any point C in a straight line AB, make ACD equal to one of the given angles (Prob. V), and DCE equal to the other; then will ECB be equal to the third angle. For the sum of the three angles at C is two right angles (Cor. 1, Theo. I); which is also the sum of the three angles of a triangle (Theo. V).

PROBLEM VIII.

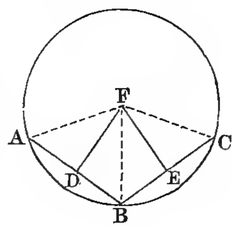
Given the hypotenuse and one side of a right-angled triangle, to construct the triangle.



Solution. Draw AB equal to the given side. At B erect a perpendicular BC (Prob. II). From A as a center, with a radius equal to the given hypotenuse, describe an arc intersecting BC at D. Join AD. It is evident that ABD is the triangle required.

PROBLEM IX.

To draw a circle through three given points.



Solution. Let A, B, and C be the given points. Draw the straight lines AB and BC, and from their middle points erect the perpendiculars DF and EF. Also, join AF, BF, and CF.

Now, because the triangles AFD, BFD, have the side AD equal to the side BD, and the side DF common, and the included angle ADF equal to the included angle BDF, the third side AF is equal to the third side BF (Theo. VII). In the same manner it may be shown that BF is equal to CF. Therefore, the three lines AF, BF, and CF are equal; and if from F as a center, with one of these equals as radius, a circle be drawn, it will pass through the three points A, B, and C.

Cor. Hence, the center of a given circle may be found by erecting perpendiculars on the middle points of two chords, and producing them till they meet.

EXERCISES.

1. Given the three sides of a triangle, to construct the triangle.

2. Given two angles and the included side, to construct the triangle.

3. Given two adjacent sides and the included angle of a parallelogram, to construct the parallelogram.

4. To draw a diameter of a given circle.

5. At a given point in the circumference of a circle, to draw a tangent to the circle.

PLANE GEOMETRY.

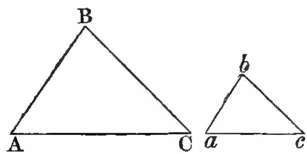
BOOK II.

PROPORTIONS OF MAGNITUDES LYING IN THE SAME PLANE.

SEC. X.—OF POLYGONS.

DEFINITIONS.

1. In mutually equiangular figures, the sides which are similarly situated with respect to the equal angles, are called *homologous* sides. Thus,



If the angles A, B, C, are respectively equal to the angles *a*, *b*, *c*, the side AB is homologous to the side *ab*, the side BC to the side *bc*, and CA to *ca*.

Two triangles or other polygons are called *similar* when they are mutually equiangular, and their homologous sides are proportional.

2. Magnitudes are called *proportionals* when the first has the same ratio to the second, that the third has to the fourth, the fifth to the sixth, etc.

The first terms of the several equal ratios are called the *antecedents*, and the second the *consequents*. Thus,

If A : B : : C : D : : E : F, the terms A, C, E, are the antecedents, and B, D, F, the consequents.

If four magnitudes are in proportion, the first and fourth terms are called the *extremes*, and the second and third the *means*. In this case the last term is called a *fourth proportional* to the three others.

If the first of three magnitudes is to the second as the second is to the third, the second is called a *mean proportional* between the two others.

It does not belong to geometry to develop the principles of proportion; but, inasmuch as they are applied in it, some of the more important of them are stated here, in explanation of the terms and phrases by which they will be quoted. None will be referred to that are not found in common arithmetics.

3. *Multiplying extremes and means*, is the phrase used in citing the principle that the product of the extremes of a proportion is equal to the product of the means. Thus,

$$\text{If } A : B :: C : D, \text{ then, } A \times D = B \times C.$$

4. *Resolving into a proportion*, refers to the principle that if the product of two magnitudes is equal to the product of two others, either couple may be made the extremes of a proportion and the other the means. Thus,

$$\text{If } A \times D = B \times C, \text{ then } A : B :: C : D.$$

5. *Multiplying an extreme and a mean equally*, refers to the principle that we may multiply an extreme and a mean by the same quantity without destroying the proportion. Thus,

$$\text{If } A : B :: C : D, \text{ then } A : 2B :: C : 2D.$$

6. *Dividing an extreme and a mean equally*, refers to the principle that we may divide an extreme and a

mean by the same quantity without destroying the proportion. Thus,

$$\text{If } A : B :: C : D, \text{ then } A : B :: \frac{C}{2} : \frac{D}{2}.$$

7. *By composition*, implies that if any number of magnitudes are proportionals, the sum of all the antecedents is to the sum of all the consequents as any one antecedent is to its consequent. Thus,

$$\text{If } A : B :: C : D :: E : F,$$
$$\text{Then } A+C+E : B+D+F :: A : B.$$

8. *By alternation*, implies that the order of the means in a proportion may be inverted. Thus,

$$\text{If } A : B :: C : D, \text{ then } A : C :: B : D.$$

9. *By equality of ratios*, implies that if two ratios are respectively equal to a third, they are equal to each other. Thus,

$$\text{If } A : B :: C : D \text{ and } A : B :: E : F,$$
$$\text{Then } C : D :: E : F.$$

10. *Multiplying corresponding terms*, is the phrase used in applying the principle that the product of the first terms of two proportions is to the product of the second terms, as the product of the third is to the product of the fourth. Thus,

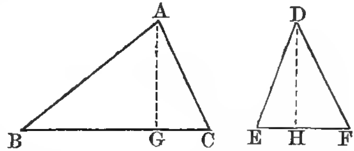
$$\text{If } A : B :: C : D, \text{ and } E : F :: G : H,$$
$$\text{Then } A \times E : B \times F :: C \times G : D \times H.$$

11. *Squaring all the terms*, is a reference to the principle that if any number of magnitudes are proportionals their squares are also proportionals. Thus,

$$\text{If } A : B :: C : D, \text{ then } A^2 : B^2 :: C^2 : D^2.$$

THEOREM I.

If two triangles have the same altitude, their areas are to each other as their bases.



Let the triangles ABC, DEF, have the same altitude; that is, let the perpendiculars AG, DH, be equal. It is to be proved that the area ABC is to the area DEF, as the base BC is to the base EF.

The area ABC is equal to $\frac{1}{2}$AG$\times$BC (Cor. 2, Theo. XVII, Book I); and the area DEF is equal to $\frac{1}{2}$DH$\times$EF. Therefore,

$$ABC : DEF : : \tfrac{1}{2}AG \times BC : \tfrac{1}{2}DH \times EF.$$

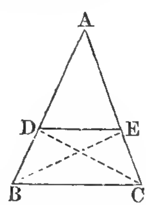
But, from the hypothesis, $\frac{1}{2}$AG is equal to $\frac{1}{2}$DH. Hence, dividing a mean and an extreme equally (Def. 6, Sec. X), we have

$$ABC : DEF : : BC : EF.$$

Therefore, if two triangles, etc.

THEOREM II.

If a straight line be drawn parallel to the base of a triangle it will cut the other sides proportionally.



In the triangle ABC, let DE be drawn parallel to the base BC. It is to be proved that it cuts AB and AC proportionally.

Join BE and CD. Now, since the triangles ADE, BDE, have the same altitude, namely, the perpendicular distance from E

to the line of their bases AB (Def. 4, Sec. V, B. I), their areas are to each other as their bases (Theo. I); that is,

ADE : DBE : : AD : DB.

Also, because the triangles ADE, CDE, have the same altitude,

ADE : CDE : : AE : EC.

But the first ratio is the same in both proportions; for ADE is common, and the triangles DBE, CDE, having the same base, DE, and the same altitude, namely, the perpendicular distance between the parallels DE and BC, are equivalent (Theo. I).

Therefore, by equality of ratios (Def. 9, Sec. X), we have

AD : DB : : AE : EC;

Or, by alternation (Def. 8, Sec. X),

AD : AE : : DB : EC.

Hence, if a straight line, etc.

Cor. 1. By composition (Def. 7, Sec. X), we get from the last proportion

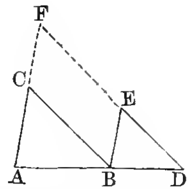
AD+DB : AE+EC : : AD : AE,

that is, AB : AC : : AD : AE.

Cor. 2. If DE be not parallel to BC, but takes some other direction, as DF, it is evident that it will not cut the other sides proportionally.

THEOREM III.

Two triangles which are mutually equiangular are also similar.



Let ABC, BDE, be two triangles having the angles at A, B, and C, respectively equal to the angles at B, D, and E. It is to be proved that these triangles are similar.

Let them be so placed that their homologous sides AB, BD (Def. 1, Sec. X), shall form one straight line. Produce AC and DE till they meet. Then, because the angle DBE is by hypothesis equal to the corresponding inner angle BAC, the lines BE and AC are parallel (Cor. 1, Theo. III, B. I); and because the angle ABC is equal to the corresponding inner angle BDE, the lines BC and DE are parallel; hence, CBEF is a parallelogram. Now, in the triangle ADF we have (Theo. II)

$$DB : BA :: DE : EF.$$

But EF is equal to BC (Theo. XIII, B. I). Hence,

$$DB : BA :: DE : BC.$$

By a like construction it may be proved in the same way that the other homologous sides are proportional. The two triangles are consequently similar (Def. 1, Sec. X).

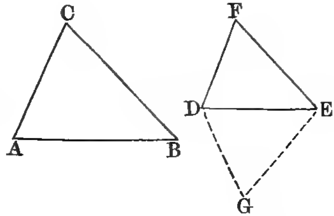
Therefore, two triangles, etc.

Cor. If two triangles have two angles of the one respectively equal to two angles of the other, they are similar; for in that case the third angles are necessarily equal (Theo. V, B. I).

THEOREM IV.

Two triangles having the sides of the one successively proportional to the sides of the other, are similar.

Let ABC, DEF, be two triangles having AB : DE : : BC : EF : : CA : FD. It may be shown that they are similar.



On DE describe the triangle DGE, having the angles EDG, DEG, respectively equal to the angles A and B. Then will the triangles ABC and DEG be similar (Cor., Theo. III); and we shall have (Def. 1, Sec. X)

AB : DE : : BC : EG.

But by hypothesis we have

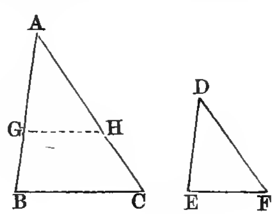
AB : DE : : BC : EF.

Hence, EG is equal to EF. In the same manner it may be shown that DG is equal to DF. Therefore, the two triangles DGE, DFE, are mutually equilateral and consequently equal (Cor., Theo. XII, B. I). But ABC is by construction similar to DGE: hence, it is also similar to DFE.

Therefore, two triangles, etc.

THEOREM V.

Two triangles having an angle of the one equal to an angle of the other, and the sides about those angles proportional, are similar.



Let the two triangles ABC, DEF, have the angle A equal to the angle D, and the sides AB, AC, proportional to

the sides DE, DF. Then will these triangles be similar.

Take AG equal to DE, and AH to DF; also, join GH. Then the triangles AGH, DEF, having two sides and the included angle of the one equal to two sides and the included angle of the other, are equal throughout (Theo. VII, B. I). Now, by hypothesis,

AB : DE : : AC : DF.

Therefore, AB : AG : : AC : AH.

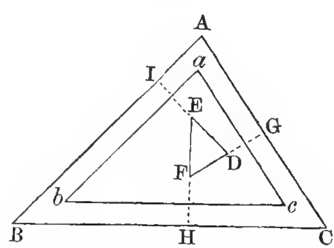
Hence, it follows that GH is parallel to BC (Cor. 2, Theo. II). Consequently the angle AGH is equal to the angle ABC (Theo. III, B. I), and AHG to ACB. Consequently, also (Cor., Theo. III), the triangle AGH or its equal DEF is similar to ABC.

Therefore, two triangles, etc.

Cor. If a triangle, as AGH, be cut off from another triangle, ABC, by a straight line parallel to the base, the two triangles will be similar.

THEOREM VI.

Two triangles having their sides respectively parallel or perpendicular to each other, are similar.



First, let the triangles ABC, *abc*, have their sides respectively parallel. Then, because the sides containing the angle A are parallel to the sides containing the angle *a*, these angles are equal (Cor. 2, Theo. III,

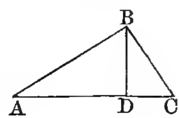
B. I); and for the same reason the angles B and b are equal, also C and c. Hence, the two triangles are mutually equiangular, and consequently similar (Theo. III).

Secondly, let the triangles ABC, DEF, have their sides respectively perpendicular to each other. Produce the sides of DEF to the points G, H, I. Now, the sum of all the angles of the quadrilateral AIDG is equal to four right angles (Theo. XX, B. I). But AID and AGD are right angles by hypothesis. Hence, IDG and IAG must be together equal to two right angles. But IDG and EDF are also together equal to two right angles (Theo. I, B. I). Therefore, subtracting equals from equals, we have the angle IAG equal to the angle EDF. In the same manner it may be proved that the angle ACB is equal to the angle DFE, and CBA to FED. Hence, the two triangles are mutually equiangular, and consequently similar.

Therefore, two triangles, etc.

THEOREM VII.

A perpendicular let fall from the right angle upon the hypotenuse of a right-angled triangle divides it into two triangles similar to the whole and to each other.



Let ABC be a triangle right-angled at B, and let BD be perpendicular to the hypotenuse AC. It is to be proved that ABD and BCD are similar to ABC and to each other.

The triangles ABD, ABC, have a right angle in

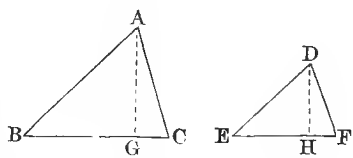
each, and the angle A common; hence, they are similar (Cor., Theo. III). The triangles BCD, ABC, have also a right angle in each, and the angle C common; and hence they are similar. Therefore, also, the triangles ABD, BCD, being both similar to the same triangle, are similar to each other.

Hence, a perpendicular, etc.

Cor. By similarity of the triangles ADB, BDC, we have AD : DB : : DB : DC; that is, *the perpendicular from the right angle is a mean proportional between the parts of the hypotenuse* (Def. 2, Sec. X).

THEOREM VIII.

The areas of similar triangles are to each other as the squares described on their homologous sides.



Let ABC, DEF, be two similar triangles of which the angles A and B are respectively equal to the angles D and E. It may be shown that their areas are to each other as AB^2 is to DE^2 (Def. 1, Sec. X).

Draw the perpendiculars AG and DH. Then, the triangles ABG, DEH, having a right angle in each, and the angles B and E equal, are similar (Cor., Theo. III), and we have

GA : HD : : AB : DE.

But by similarity of ABC and DEF we have

BC : EF : : AB : DE.

Multiplying corresponding terms (Def. 10, Sec. X) and dividing an extreme and a mean equally (Def. 6, Sec. X),

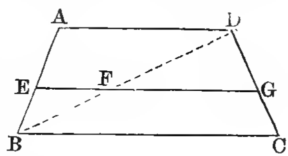
$$\frac{BC \times GA}{2} : \frac{EF \times HD}{2} : : AB^2 : DE^2.$$

But the first two terms of this proportion represent the areas of the triangles ABC and DEF (Cor. 2, Theo. XVII, B. I).

Therefore, the areas, etc.

THEOREM IX.

If between the two parallel sides of a trapezoid a third parallel be drawn, it will cut the other sides proportionally.



In the trapezoid ABCD, let EG be a third parallel to AD and BC. Then will BE be to EA as CG is to GD.

Draw the diagonal BD. Now, because EF is parallel to the base AD of the triangle BAD, we have (Theo. II)

$$BE : EA : : BF : FD;$$

and, because FG is parallel to the base BC of the triangle DBC,

$$CG : GD : : BF : FD.$$

Therefore, by equality of ratios (Def. 9, Sec. X),

$$BE : EA : : CG : GD.$$

Hence, if between, etc.

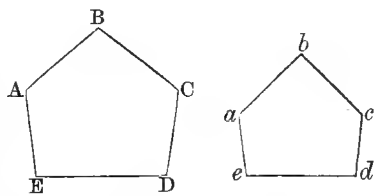
Cor. 1. *If the third parallel bisects the oblique sides, it is equal to one-half the sum of the parallel sides.*

For by similarity of the triangles BEF, BAD (Cor. Theo. V), if BE be one-half of BA, EF will be one-half of AD; and in the same manner it may be shown that FG will be one-half of BC: consequently, the whole line EG will be one-half the sum of AD and BC.

Cor. 2. The area of the trapezoid (Theo. XVIII, B. I) is equal to its altitude multiplied by EG, which is its mean breadth.

THEOREM X.

The perimeters of similar polygons are to each other as their homologous sides.



Let ABCDE, *abcde*, be two similar polygons, having the angles A, B, C, etc., respectively equal to the angles *a*, *b*, *c*, etc. Then will their perimeters be to each other as any two homologous sides AB and *ab*.

By similarity of the two polygons (Def. 1, Sec. X) we have

$$AB : ab :: BC : bc :: CD : cd :: DE : de :: EA : ea.$$

And therefore, by composition (Def. 7, Sec. X),

$$AB+BC+CD+DE+EA : ab+bc+cd+de+ea :: AB : ab.$$

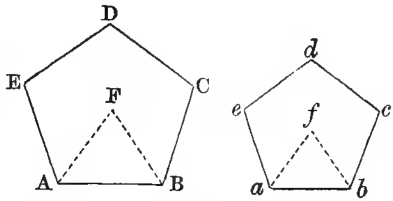
But the first two terms of the last proportion represent the perimeters of the two polygons (Def. 3, Sec. VII, B. I.)

Therefore, the perimeters, etc.

THEOREM XI.

The areas of two regular polygons of the same number of sides are to each other as the squares of their sides.

Let ABCDE, *abcde*, be two regular polygons of the same number of sides, for example, five. The area of the former is to the area of the latter as AB^2 is to ab^2.



Since the two polygons have the same number of sides and angles, it is evident (Theo. XX, B. I) that the sum of all the angles of ABCDE is equal to the sum of all the angles of *abcde*, and consequently that each angle of ABCDE is equal to each angle of *abcde* (Def. 2, Sec. VII).

Bisect the angles A and B by AF and BF, and the angles a and b by af and bf. Now, since the triangles ABF, abf, have two angles of the one equal to two angles of the other, they are similar (Cor., Theo. III); hence,

$$ABF : abf :: AB^2 : ab^2 \text{ (Theo. VIII).}$$

Multiplying an extreme and a mean equally,

$$5\,ABF : 5\,abf :: AB^2 : ab^2 \text{ (Def. 5, Sec. X).}$$

But five times ABF is the area of ABCDE (Theo. XXI, B. I), and five times abf is the area of *abcde*.

Therefore, the areas, etc.

EXERCISES.

1. Prove that two parallelograms of the same altitude are to each other as their bases.

2. Prove that if two isosceles triangles have their vertical angles equal they are similar.

3. If the area of a regular pentagon whose side is 1 be 1.72, what is the area of a regular pentagon whose side is 5?

4. In two similar polygons, if a side of one be 7, and the homologous side of the other 13, and if the perimeter of the former be 39, what is the perimeter of the latter?

SEC. XI.—OF CIRCLES.

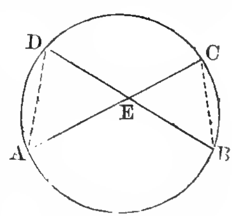
DEFINITIONS.

1. Two ARCS of circles are called *similar* when they are equal parts of the circumferences to which they belong.

2. Two SECTORS are called *similar* when their arcs are similar.

THEOREM XII.

If two chords intersect in a circle, the product of the parts of the one is equal to the product of the parts of the other.



In the circle ABCD let the chords AC, BD, intersect each other in E. It is to be proved that AE×EC=BE×ED.

Join AD and BC. Now, in the triangles ADE, BCE, the vertical angles AED and BEC are equal (Theo. II,

B. I); also, the angles A and B are equal, being both measured by half the same arc, DC (Theo. XXVI, B. I); hence, the third angles are equal, and the two triangles are similar (Theo. III), and we have

$$AE : BE :: ED : EC.$$

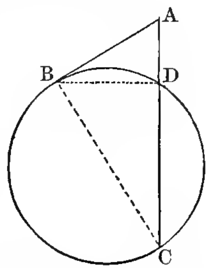
Then, multiplying extremes and means, we get

$$AE \times EC = BE \times ED.$$

Therefore, if two chords, etc.

THEOREM XIII.

If from a point without a circle a tangent and a secant be drawn, the tangent will be a mean proportional between the secant and its external part.



Let AB be a tangent and AC a secant to a circle. Then will

$$AC : AB :: AB : AD.$$

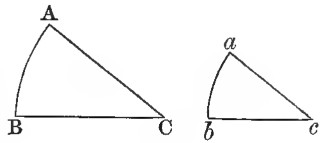
Join BC and BD. Now, the angle DBA, contained by a tangent and chord, is measured by half the arc DB (Theo. XXVII, B. I); and the angle C, being an inscribed angle, is measured by half the same arc (Theo. XXVI, B. I); therefore, these two angles are equal. But the angle A is common to the two triangles BAD and BAC. Hence, these triangles have two angles of the one equal to two angles of the other, and are consequently similar (Cor., Theo. III).

Therefore, $AC : AB :: AB : AD.$

That is, if from a point, etc.

THEOREM XIV.

Similar arcs in different circles subtend equal angles at the centers.



Let AB and *ab* be similar arcs in two circles whose centers are C and *c*. It may be shown that the angles at C and *c* are equal.

Whatever portion AB is of the circumference to which it belongs, the angle C is the same portion of four right angles (Schol., Theo. XXIV, B. I); and whatever portion *ab* is of the other circumference, the angle *c* is the same portion of four right angles. But AB is the same portion of the first circumference that *ab* is of the second (Def. 1, Sec. XI). Hence, the angles C and *c* are equal portions of four right angles; hence, also, they are equal to each other.

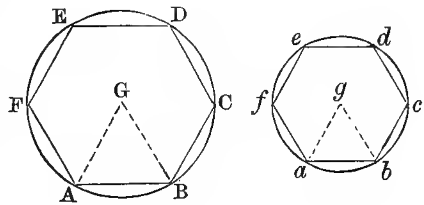
Therefore, similar arcs, etc.

Cor. The sectors ABC, *abc*, are similar (Def. 2, Sec. XI). Hence, the sides of similar sectors contain equal angles. Hence, also, similar sectors are contained an equal number of times in the circles to which they belong.

Schol. Since a degree is $\frac{1}{360}$ of the circumference in which it is taken (Schol., Theo. XXIV, B. I), it follows that degrees in unequal circles are similar but not equal arcs.

THEOREM XV.

The circumferences of two circles are to each other as their diameters.



Let ACE and *ace* be two circles having two similar regular polygons inscribed, one in each. Draw the radii AG, BG; *ag*, *bg*. Now, the angles G and *g*, being subtended by similar arcs, AB and *ab*, are equal (Theo. XIV); and the sides containing them are by construction proportional; therefore, the two triangles AGB, *agb*, are similar (Theo. V), and we have

$$AB : ab :: AG : ag :: 2\,AG : 2\,ag.$$

But the perimeter ABCDEF is to the perimeter *abcdef*, as AB is to *ab* (Theo. X), and consequently as 2 AG is to 2 *ag*. Now, if the number of sides of the similar polygons be indefinitely increased, the perimeters will ultimately coincide with the circumferences. Therefore,

Circumference ACE : circumference *ace* : : 2 AG : 2 *ag*.

But 2 AG and 2 *ag* represent the diameters of the two circles (Def. 2, Sec. 8, B. I).

Therefore, the circumferences, etc.

Cor. 1. Representing the diameter of any circle by D, and its circumference by C, and remembering that the circumference of a circle whose diameter is 1 is 3.14159 (Schol., Theo. XXX, B. I), we have, by the above theorem,

$$1 : D :: 3.14159 : C.$$

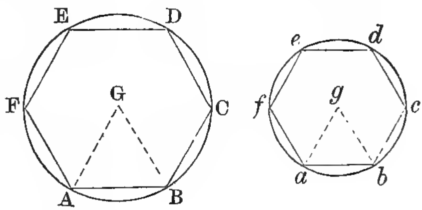
Multiplying extremes and means, we get $C = 3.14159 \times D$. That is,

The circumference of any circle is equal to its diameter multiplied by 3.14159.

Cor. 2. The diameter is equal to the circumference divided by 3.14159.

THEOREM XVI.

The areas of two circles are to each other as the squares of their diameters.



The same construction being used as in the last theorem, it may be shown as before, that

$$AB : ab : : 2\,AG : 2\,ag.$$

Hence, by squaring all the terms (Def. 11, Sec. X),

$$AB^2 : ab^2 : : (2\,AG)^2 : (2\,ag)^2.$$

But the area of the polygon ABCDEF is to the area of the polygon *abcdef*, as AB^2 is to ab^2 (Theo. XI), and, consequently, as $(2\,AG)^2$ is to $(2\,ag)^2$. Now, if the number of sides of the two polygons be indefinitely increased, their areas will ultimately coincide with the areas of the circles. Hence, the area of the circle ACE is to the area of the circle *ace* as $(2\,AG)^2$ is to $(2\,ag)^2$.

That is, the areas of two circles, etc.

Cor. Representing the diameter of any circle by D and its area by A, and remembering that the area

of a circle whose diameter is 1 is .7854 (Cor., Theo. XXX, B. I), we have

$$1^2 : D^2 :: .7854 : A.$$

But 1^2 is 1. Therefore, multiplying extremes and means, we get $A = .7854 \times D^2$. That is,

The area of any circle is equal to the square of its diameter multiplied by .7854.

EXERCISES.

1. If a secant to a circle be 12 feet, and its external part 3 feet, what will be the length of a tangent to the circle, drawn from the same point?

2. If the diameter of a circle be 24, what will be the length of its circumference?

3. If the circumference of a circle be 3924, what is the length of its diameter?

4. The diameter of a circle is 16: what its area?

5. Prove that the areas of similar sectors are to each other as the squares of their radii.

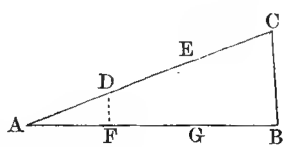
6. Prove that the area of a circle equals the square of its radius multiplied by 3.14159.

SEC. XII.—PROBLEMS IN CONSTRUCTION.

PROBLEM I.

To divide a straight line into any number of equal parts.

Solution. Let AB be the given straight line, which it is required to divide into a certain number of equal parts, for example, three.

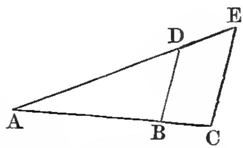


From A draw any straight line AD; and lay off DE and EC in the same direction, each equal to AD. Join CB, and through D draw DF parallel to CB. Now, AD : AC : : AF : AB (Cor. 1, Theo. II). But AD is one-third of AC, by construction; therefore, AF is one-third of AB. Now, lay off FG equal to AF, and it is evident that AB will be divided into three equal parts.

PROBLEM II.

To find a fourth proportional to three given straight lines.

Solution. Draw two straight lines containing any angle A. Make AB, BC, and AD respectively equal to the three given lines, and join BD.

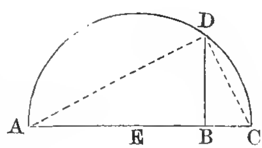


Through C draw CE parallel to BD, and produce AD to meet it in E. Now, AB : BC : : AD : DE (Theo. II). Hence, DE is the fourth proportional required (Def. 2, Sec. X).

PROBLEM III.

To find a mean proportional between two given straight lines.

Solution. Lay off, in one straight line, AB and BC, respectively equal to the two given lines. Find E, the middle point of AC.

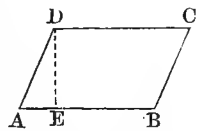


Then, from E as a center, with EA or EC as radius, describe a semicircle. At B erect BD perpendicular to AC, and join AD and DC. Now, since the angle

ADC is inscribed in a semicircle, it is a right angle (Cor. 2, Theo. XXVI, B. I). Hence, BD is a mean proportional between AB and BC (Cor., Theo. VII).

PROBLEM IV.

To describe a square that shall be equivalent to a given parallelogram.



Solution. Let ABCD be the given parallelogram. Between its base AB and its altitude ED, find a mean proportional, which denote by M. Then, since

AB : M : : M : ED, we have $AB \times ED = M^2$.

But $AB \times ED$ represents the area of the parallelogram (Cor. 1, Theo. XVII, B. I). Therefore, the square described on M is the square required.

Cor. It is evident that if we find a mean proportional between the base of a triangle and half its altitude (Cor. 2, Theo. XVII, B. I), we shall have the side of a square equivalent to the triangle.

EXERCISES.

1. To construct a triangle that shall be similar to a given triangle.

2. On a given straight line, to describe a rectangle that shall be equivalent to a given square.

BOOK III.

SOLID GEOMETRY.

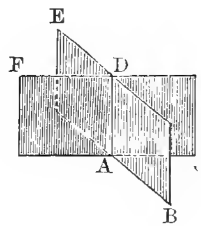
SEC. XIII.—PLANES AND THEIR INCLINATIONS.

DEFINITIONS.

1. A straight line is said to be *perpendicular to a plane* when it is perpendicular to every straight line it can meet in that plane.

2. The line in which two planes cut one another is called their INTERSECTION.

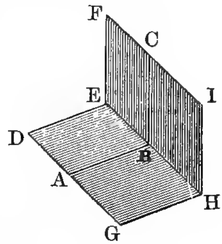
Schol. The intersection of two planes is a straight line. For, if A and D are any two points in the intersection of the planes FC and EB, the straight line joining these points must be in both planes (Def. 8, Sec. I); it is, therefore, their intersection.



3. The *angle of inclination* of two planes is that contained by any two straight lines perpendicular to their intersection, one in each plane.

If the line in each plane is perpendicular to the other plane, the two planes are said to be *perpendicular* to each other, the angle of inclination being a right angle.

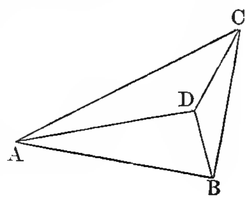
Thus, if AB and CB are both perpendicular to the intersection EH, then ABC is the angle of inclination of the planes DH and HF. If AB is also perpendicular to the plane HF, and CB to DH, the two planes are perpendicular to each other.



4. Two planes are said to be *parallel* with one another when their intersections by any third plane are parallel.

Cor. Two parallel planes can never meet; for, if they were to meet in any point, their intersections by a third plane passing through that point would also meet; which is contrary to the definition.

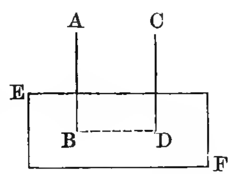
5. The divergence of three or more planes from a single point constitutes a SOLID ANGLE. Thus, A is a solid angle contained by the three planes BAC, CAD, DAB.



THEOREM I.

Two straight lines perpendicular to the same plane are parallel to each other.

Let AB and CD be two perpendiculars to the plane EF. It is to be shown that they are parallel.



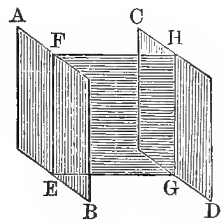
Join BD. Now, AB and CD, being perpendicular to the plane, are also perpendicular to the line BD

which they meet in it (Def. 1). Also, a plane perpendicular to EF and passing through BD will contain both AB and CD (Def. 3). But straight lines in the same plane and perpendicular to the same straight line are parallel (Cor. 2, Theo. IV, B. I). Therefore, AB is parallel to CD.

That is, two straight lines, etc.

THEOREM II.

Parallel lines intercepted between parallel planes are equal.



Let FH and EG be any two parallel lines intercepted between two parallel planes, AB and CD. It is to be shown that FH is equal to EG.

Through the parallels FH and EG let the plane FG pass. Its intersections with the parallel planes AB and CD will be parallel lines (Def. 4); that is, FE is parallel to HG. Hence, FEGH is a parallelogram, and FH is equal to EG (Theo. XIII, B. I).

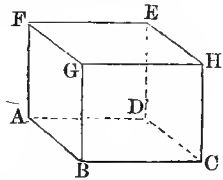
Therefore, parallel lines, etc.

Cor. 1. *Any straight line*, as FH, *perpendicular to one of two parallel planes*, as CD, *will be perpendicular to the other.* For FHG being a right angle (Def. 1), HFE will also be a right angle (Cor. 1, Theo. IV, B. I), whatever may be the position of the parallels FE and HG in their planes.

Cor. 2. The perpendicular distance between two parallel planes is everywhere the same.

SEC. XIV.—OF PARALLELOPIPEDS.

1. A PARALLELOPIPED is a solid bounded by six plane faces, of which each one is parallel to its opposite. The intersections of the faces are called *edges*.



Cor. Any face of a parallelopiped is a parallelogram. For the lines GH and BC, being the intersections of two parallel planes, FH and AC, with a third plane GC, are parallel (Def. 4, Sec. XIII); and in the same way it may be shown that BG and CH are parallel; hence, BCHG is a parallelogram (Def. 2, Sec. VI, B. I).

2. A *right* PARALLELOPIPED is one whose edges are perpendicular to the faces. Any other parallelopiped is called *oblique*.

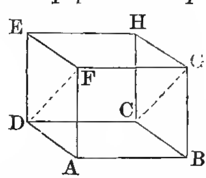
3. A CUBE is a right parallelopiped whose length, breadth, and hight are equal.

4. Two parallelopipeds, or other solids, are called *equivalent* when they contain the same amount of solid space.

THEOREM III.

Any two opposite faces of a parallelopiped are equal.

Let AH be a parallelopiped. It is to be proved that any two opposite faces, as DEFA, CHGB, are equal. Join DF and CG. Now, since EFGH is a parallelogram (Cor. Def. 1), EF is equal to HG (Theo. XIII, B. I); and in the same manner it may be shown that DE is equal to CH. But FG and DC,



being both equal and parallel to EH (Theo. XIII, B. I), must be equal and parallel to each other. Hence, DCGF is a parallelogram (Theo. XIV, B. I), and DF is equal CG. Wherefore, the triangles DEF, CHG, are mutually equilateral, and consequently equal. But the former triangle is half the parallelogram DEFA (Cor. 1, Theo. XIII, B. I), and the latter triangle is half the parallelogram CHGB: hence, these two parallelograms are equal.

Therefore, any two opposite faces, etc.

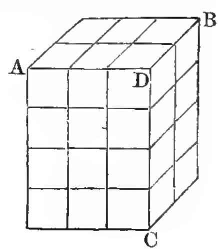
Schol. Any face of a parallelopiped may be taken as the *base.* We may designate DABC and EFGH as the *lower and upper bases.* The perpendicular distance between the bases is called the *hight* or *altitude.*

Cor. 1. If EB be a right parallelopiped, the edges AF and BG, since, by definition, they are both perpendicular to the face DB, must be perpendicular also to the straight line AB, which they meet in that plane (Def. 1, Sec. XIII). Hence, ABGF, or *any face of a right parallelopiped, is a rectangle.*

Cor. 2. The faces of a cube are all equal squares.

THEOREM IV.

The solidity of a right parallelopiped is equal to the area of its base multiplied by its hight.



Let ABC be a right parallelopiped. It is to be proved that its solidity is equal to the area of its base AB, multiplied by its hight DC.

Let planes pass through the solid parallel to the three faces

AB, BC, CA, dividing the edges DA, DB, DC, into parts of equal length. It is evident that the parallelopiped will thus be divided into a number of equal cubes. Let one of these be taken as the unit of solidity. Now, in the layer next to the base, there will be as many of these solid units as there are corresponding units of area in the base; and there will be as many equal layers as there are corresponding linear units in the hight.

Therefore, the solidity of a right parallelopiped, etc.

Schol. If the edges are incommensurable, so that no unit can be found into which they can all be divided without remainder, the above theorem will still hold true; for by taking the unit smaller and smaller, the remainder can be made less than any assignable quantity.

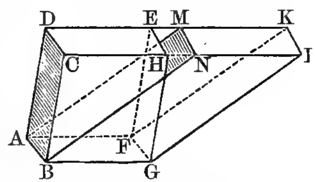
When the linear unit is one inch, the unit of solidity is a cubic inch; when the linear unit is one foot, the unit of solidity is a cubic foot, etc.

Cor. 1. The solidity of a right parallelopiped is equal to the product of its length, breadth, and hight.

Cor. 2. The solidity of a cube may be found by raising one of its edges to the third power.

THEOREM V.

Parallelopipeds on the same base and between the same parallels are equivalent.

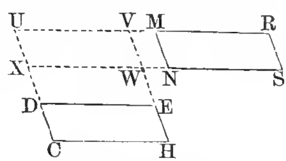


Let AGHD and AGLM be two parallelopipeds on the same base AG, and between the same parallels BG, CL; AF, DK. It may be shown that they are equivalent.

The parallelograms AC, FH, being opposite faces of a parallelopiped, are equal (Theo. III). Also, the parallelograms DH, ML, being both equal to AG, are equal to each other; hence, if we add EN to both, the parallelograms DN and EL must be equal. It is also evident that the side CN is equal to the side HL. But CB is equal to HG, and the angle BCN is equal to the corresponding outer angle GHL; therefore, the triangles BCN, GHL, having two sides and the included angle of the one equal to two sides and the included angle of the other, are equal (Theo. VII, B. I).

If now the solid FGLE be applied to the solid ABND, so that the face EL shall coincide with its equal DN, then the faces GHL and BCN, being equal and in the same plane, will coincide; and in the same manner it may be shown that the faces FEK and ADM will coincide; hence, also, the face FH will coincide with the face AC, and FL with AN; and the two solids are consequently equal. If, then, from the whole solid AL, we take away these equal portions in turn, the remainder AGHD will be equivalent to the remainder AGLM.

Therefore, parallelopipeds on the same base, etc.



Schol. Any two parallelopipeds on the same base and between the same planes are equivalent. For, if their upper bases are not between the same parallels, but have some other position, as DH and MS, produce two sides of each till they intersect as in the figure; then UVWX

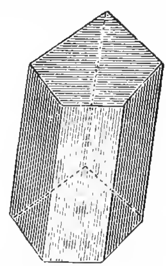
can be the upper base of a parallelopiped on the same lower base and between the same parallels with both the other parallelopipeds; hence, by the above demonstration, both the others will be equivalent to this third parallelopiped; they will, therefore, be equivalent to each other.

Cor. The solidity of any parallelopiped on a rectangular base, is equal to the area of its base multiplied by its altitude; for it is equivalent to a right parallelopiped of the same base and altitude.

SEC. XV.—THE PRISM AND THE CYLINDER.

DEFINITIONS.

1. A PRISM is a solid bounded by plane faces, of which two are equal and parallel polygons, and the others parallelograms. It includes the parallelopiped, as one species.

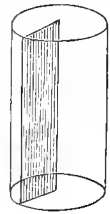


The equal and parallel polygons are called the BASES. The other faces together form the CONVEX or LATERAL SURFACE. The edges joining the corresponding angles of the two bases are called the *principal edges.*

A prism is called triangular, quadrangular, pentagonal, etc., according as its base is a triangle, a quadrilateral, a pentagon, etc.

2. A *right* PRISM is one whose principal edges are perpendicular to the bases. Any other prism is called *oblique.*

3. A CYLINDER is a solid described by the revolution of a rectangle about one side, which remains fixed.



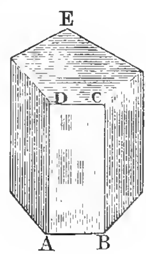
The fixed side is called the AXIS of the cylinder. The opposite side describes the CONVEX SURFACE. The circles described by two other sides are called the BASES.

4. A prism is said to be INSCRIBED in a cylinder when its bases are inscribed in the bases of the cylinder, and its principal edges lie in the convex surface of the cylinder.

5. The HIGHT, or ALTITUDE, of a prism or a cylinder, is the perpendicular distance between the planes of its bases.

THEOREM VI.

The convex surface of a right prism is equal to the perimeter of its base multiplied by its hight.



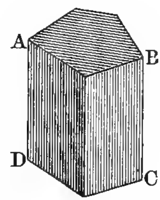
Let ABE be a right prism. Since its principal edges are by definition perpendicular to the bases, any one of them may be taken as the hight of the prism. Now, AD and BC, being both perpendicular to the lower base, are also perpendicular to AB, which they meet in the plane of that base (Def. 1, Sec. XIII); hence, the parallelogram ABCD is a rectangle, and its area is equal to its base AB multiplied by its altitude BC (Cor. 1, Theo. XVII, B. I). In the same manner it may be shown that the area of each of the other faces composing the convex surface is equal to its base multi-

plied by the altitude of the prism. Therefore, the sum of their areas is equal to the sum of their bases multiplied by the common altitude. But the sum of their bases constitutes the perimeter of the base of the prism.

Hence, the convex surface, etc.

THEOREM VII.

The solidity of any prism is equal to the area of its base multiplied by its altitude.

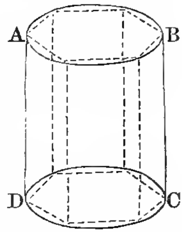


Let ABCD be any prism. Now, whatever may be the form of its base, AB, it is evident that it may be divided into an indefinitely large number of small rectangles, so that the remainder, if there be any, shall be less than any assignable quantity; and the base DC may be divided into the same number of equal rectangles having their sides respectively parallel to the sides of the others. Now, each rectangle in the lower base, and its corresponding rectangle in the upper base, may be considered as the opposite bases of a small parallelopiped; and the prism will be made up of such parallelopipeds. But the solidity of each of these will be equal to the area of its base multiplied by its altitude (Cor., Theo. V), which is the same as the altitude of the prism: hence, the sum of their solidities will be equal to the sum of their bases multiplied by their common altitude.

That is, the solidity of any prism is equal, etc.

THEOREM VIII.

The convex surface of a cylinder is equal to the circumference of its base multiplied by its altitude; and its solidity is equal to the area of its base multiplied by its altitude.

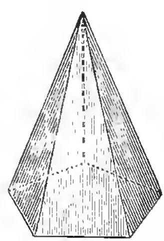


Let ABCD be a cylinder, having a prism inscribed in it whose base is a regular polygon. Now, if the number of sides of this polygon be indefinitely increased, its perimeter will ultimately coincide with the circumference of the base of the cylinder. Then, also, the convex surface of the prism will coincide with the convex surface of the cylinder, and the solidity of the prism with the solidity of the cylinder. But the convex surface of the prism is equal to the perimeter of the base multiplied by the altitude (Theo. VI), and its solidity is equal to the area of the base multiplied by the altitude (Theo. VII).

Therefore, the convex surface of a cylinder, etc.

SEC. XVI.—PYRAMIDS AND CONES.

DEFINITIONS.



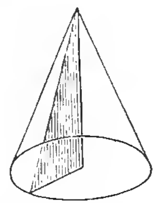
1. A PYRAMID is a solid bounded by plane faces, of which one is any polygon, and the others triangles having a common vertex. The polygon is called the *base*. The triangles together form the *convex* or *lateral* surface.

Pyramids are called triangular, quadrangular, pen-

tagonal, etc., according as their bases are triangles, quadrilaterals, pentagons, etc.

2. A *regular* PYRAMID is one whose base is a regular polygon, and the triangular faces are equal and isosceles.

3. A CONE is a solid described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed. The fixed side is called the *axis* of the cone. The hypotenuse describes the *convex surface*. The circle described by the other revolving side is called the *base*.



4. The ALTITUDE of a pyramid or cone is the perpendicular distance from the vertex to the plane of the base.

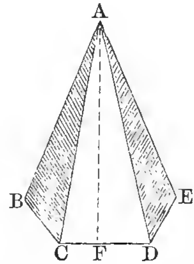
5. The *slant hight* of a regular pyramid is the perpendicular let fall from the vertex upon the base of any one of its triangular faces. The *side* or *slant hight* of a cone is the straight line drawn from the vertex to any point in the circumference of the base.

6. A FRUSTUM of a pyramid or a cone, is the portion next the base cut off by a plane parallel to the base. The slant hight of a frustum is that part of the slant hight of the whole solid which lies on the frustum.

7. A SECTION of any solid is the surface in which it is divided by a plane which passes through it.

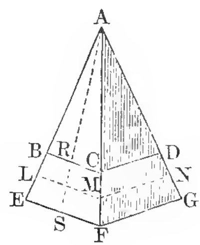
THEOREM IX.

The convex surface of a regular pyramid is equal to half the product of the perimeter of the base by the slant hight.



Let ABCDE be a regular pyramid, of which AF is the slant hight. The area of the triangle ACD is equal to half the product of its base CD into its altitude, which is the slant hight. Consequently the areas of all the equal triangles composing the convex surface are together equal to half the product of the sum of their bases by the slant hight. But the sum of their bases constitutes the perimeter of the base of the pyramid.

Therefore, the convex surface of a regular pyramid, etc.



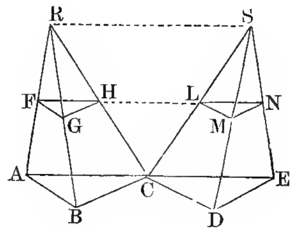
Schol. 1. The trapezoids BF, CG, etc., composing the convex surface of a frustum of a regular pyramid, are equal to each other; for they are the differences between the equal triangles AEF, AFG, etc., and the equal triangles ABC, ACD, etc.

Schol. 2. Since the area of the trapezoid BF is equal (Cor. 2, Theo. IX, B. II) to its mean breadth LM multiplied by RS, which is the slant hight of the frustum, and since the same is true of each of the other trapezoids, it follows that *the convex surface of a frustum of a regular pyramid is equal to its slant*

hight multiplied into the perimeter of a middle section between its two bases.

THEOREM X.

Two triangular pyramids of equal bases and altitudes are equivalent.



Let the two pyramids have their bases in the same plane, and DC equal to BC. Conceive a plane to cut the two solids parallel to the plane of their bases, making the triangular sections FGH and LMN. Now, since the vertices R and S are by hypothesis equidistant above the plane of the bases, a third plane may pass through these two points parallel with the other two planes (Cor. 2, Theo. II). Then the lines HL and RS, being the intersections of two parallel planes by a third plane CRS, will be parallel (Def. 4, Sec. XIII); hence, the triangles CRS, CHL, are similar (Cor., Theo. V, B. II). In the same manner it may be shown that the triangles CRB, HRG, are similar; also, CSD and LSM.

Now, CR : HR : : CB : HG.
Also, CS : LS : : CD : LM.

But by similarity of CRS and CHL, we have

CR : HR : : CS : LS.

Hence, by equality of ratios (Def. 9, Sec. X),

CB : HG : : CD : LM.

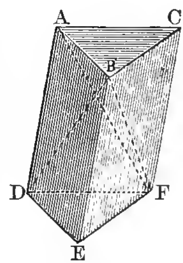
But CB is by hypothesis equal to CD; therefore, HG is equal to LM. In the same way it may be shown

that the other sides of the triangle FHG are respectively equal to the other sides of the triangle LMN; and these triangles are consequently equal throughout. Hence, at equal hights, the sections parallel to the bases are equal; and the two pyramids may be conceived to be applied to one another so as to coincide at all equal hights successively, from their bases to their vertices. They are, therefore, equivalent.

That is, two triangular pyramids, etc.

THEOREM XI.

A triangular pyramid is one-third of a triangular prism of the same base and altitude.



Let ABCDEF be a triangular prism. Join AF, BF, and BD. Now, the pyramid BDEF, cut off by the plane of the triangle BDF, is equivalent (Theo. X) to the pyramid FABC, cut off by the plane of the triangle ABF; for they have equal bases, DEF and ABC (Def. 1, Sec. XV), and the same altitude, namely the altitude of the prism. But the pyramid FABC is equivalent to the third pyramid BADF; for they have equal bases ADF and FCA (Cor. 1, Theo. XIII, B. I), and the same altitude, namely the perpendicular distance of their common vertex B above the plane of their bases ADFC. Hence, the pyramid BDEF is one-third of the prism ABCDEF.

That is, a triangular pyramid is one-third, etc.

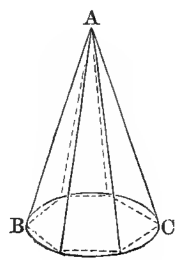
Cor. 1. Hence, the solidity of a triangular pyramid

is equal to one-third the product of its base by its altitude (Theo. VII).

Cor. 2. *The solidity of any pyramid whatever, is equal to one-third the product of its base by its altitude;* for, by dividing its base into triangles, and passing planes through the lines of division and the vertex, the pyramid can be divided into a number of triangular pyramids; and the sum of their solidities will be equal to one-third the product of the sum of their bases by their common altitude.

THEOREM XII.

The convex surface of a cone is equal to half the product of the circumference of the base by the slant hight; and its solidity is equal to one-third the product of the area of the base by the altitude.



Let ABC be a cone, having a regular pyramid inscribed in it. If the number of sides of the polygon constituting the base of the pyramid be indefinitely increased, its perimeter will ultimately coincide with the circumference of the base of the cone. Then, the slant hight of the pyramid will be equal to the slant hight of the cone, and the convex surface and solidity of the pyramid to the convex surface and solidity of the cone. But the convex surface of the pyramid will be equal to half the product of the perimeter of the base by the slant hight (Theo. IX); and the solidity of the pyramid will be equal to one-third

the product of the area of the base by the altitude (Cor. 2, Theo. XI).

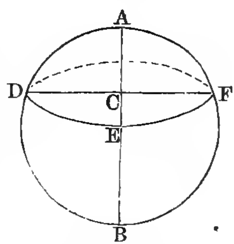
Therefore, the convex surfaces of a cone, etc.

Schol. In the same manner, it may be shown that the convex surface of a frustum of a cone is equal to the product of the slant hight into the circumference of a middle section between the two bases (Schol. 2, Theo. IX). And as this will hold true however small the upper base may be, it will hold true of the cone itself, which may be treated as a frustum whose upper base is nothing.

SEC. XVII.—THE SPHERE.

DEFINITIONS.

1. A SPHERE is a solid which may be described by the revolution of a semicircle around its diameter as a fixed axis.



The semi-circumference describes the *convex surface.* The *center* is the middle point of the axis. The *radius* is a straight line from the center to any point of the surface; and it is equal to the radius of the semicircle. A *diameter* is a double radius.

Schol. As the semicircle ADB revolves about the axis AB, a perpendicular, as DC, let fall on the axis from any point in the circumference, will describe a circle.

2. A GREAT CIRCLE on the sphere is one whose

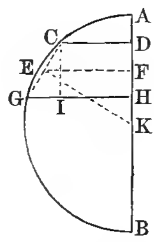
plane passes through the center. Its radius is the same as the radius of the sphere. Its circumference is also called the circumference of the sphere. Any other circle on the sphere is called a SMALL CIRCLE.

3. A SEGMENT of a sphere is any portion, as ADEF, cut off from the solid by a plane passing through it.

4. A frustum of a cone is said to be inscribed in a sphere, when the circumferences of its bases lie in the surface of the sphere.

THEOREM XIII.

If a frustum of a cone be inscribed in a sphere, its convex surface will be equal to the altitude of the frustum multiplied by the circumference of a circle, whose radius is a perpendicular from the center of the sphere to the slant hight of the frustum.



Let CD and GH be both perpendicular to the axis AB. Then, as the semicircle revolves about the axis, describing the sphere, the trapezoid DCGH will describe a frustum of a cone, which will be inscribed in the sphere. From the center K let fall the perpendicular KE, on the chord CG; then E will be the middle point of CG (Theo. XXV, B. I). Draw EF perpendicular to AB, and CI perpendicular to GH.

Now, since the triangles GCI, KEF, have the sides of the one respectively perpendicular to the sides of the other, they are similar (Theo. VI, B. II).

Hence, GC : CI : : KE : EF.

Multiplying an extreme and a mean equally (Def. 5, Sec. X),

GC : CI : : 2KE×3.14159 : 2EF×3.14159.

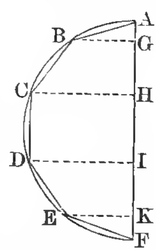
The first term of this proportion is the slant hight of the frustum; the second is its altitude; the third is the circumference of a circle whose radius is KE (Cor. 1, Theo. XV, B. II); the fourth is the circumference of a circle whose radius is EF. Therefore, multiplying extremes and means, we have the product of the slant hight into the circumference of a middle section of the frustum, equal to the product of its altitude into the circumference of a circle whose radius is the perpendicular from the center to the slant hight. But the former product is equal to the convex surface of the frustum (Schol., Theo. XII); consequently, the latter product is equal to the same.

That is, if a frustum of a cone be inscribed, etc.

Schol. If the chord CG be one side of a regular inscribed polygon, it is evident that KE will be its apothegm (Cor. 2, Theo. XXVIII, B. I).

THEOREM XIV.

The surface of a sphere is equal to the product of its diameter by its circumference.



Let ABCDEF be a semicircle, having the half of a regular polygon inscribed in it. As the semicircle revolves about the axis AF, describing the sphere, each of the trapezoids GBCH, HCDI, etc., will describe a frustum of a cone, which will be inscribed in the sphere. The convex surface

of each of these frustums will be equal to its altitude multiplied by the circumference of a circle whose radius is the apothegm of the polygon (Schol., Theo. XIII). Therefore, the sum of their convex surfaces will be equal to the circumference of such a circle multiplied by the sum of the altitudes GH, HI, etc., that is, multiplied by AF, the diameter of the sphere. Now if the number of sides of the semi-polygon be indefinitely increased, its perimeter will ultimately coincide with the semi-circumference, and its apothegm with the radius of the sphere. Then, the sum of the convex surfaces of the frustums will be equal to the surface of the sphere, and the circumference of which the apothegm is radius will be the circumference of the sphere (Def. 2).

Therefore, the surface of a sphere is equal, etc.

Cor. 1. Since the circumference is equal to $3.14159 \times D$ (Cor. 1, Theo. XV, B. II), it follows that the surface is equal to $3.14159 \times D^2$.

Cor. 2. The convex surface of any segment of a sphere, as that described by the arc ABC, is equal to its altitude AH multiplied by the circumference of the sphere.

THEOREM XV.

The solidity of a sphere is equal to its surface multiplied by one-sixth of its diameter.

The sphere may be considered as made up of infinitely small pyramids, whose bases together form the surface of the sphere, and whose common vertex is at the center. Now, the solidity of each of these pyramids will be equal (Cor. 2, Theo. XI) to the product

of its base by one-third of its altitude, that is, by one third the radius of the sphere. Hence, the sum of their solidities will be equal to the sum of their bases multiplied by one-third of the radius, or one-sixth of the diameter.

That is, the solidity of a sphere is equal, etc.

Cor. Since the surface of a sphere is equal to $3.14159 \times D^2$, it follows that its solidity is equal to $3.14159 \times D^2 \times \frac{D}{6}$, which, by reduction, becomes $.5236 \times D^3$. That is, *the solidity of a sphere is equal to the cube of its diameter multiplied by* .5236.

SUPPLEMENT.

SEC. XVIII.—MISCELLANEOUS EXAMPLES.

1. What is the side of a square inscribed in a circle whose diameter is 5 feet?

2. Prove that the side of a square is to its diagonal as 1 to the square root of 2.

3. Prove that the area of a square described about a circle is double the area of a square inscribed in the same circle.

4. What is the altitude of an equilateral triangle whose side is 12 feet?

5. If the altitude of a cone is 9 feet, and the diameter of its base 5 feet, 4 inches, what is its convex surface? Its solidity?

6. What is the surface of a sphere whose diameter is 7 feet? Its solidity?

7. If the altitude of a prism is 5 feet, and the area of its base 18 square feet, what is its solidity?

8. What is the solidity of a cube whose edge is 6 feet? Its surface?

9. What is the lateral surface of a regular pyramid whose slant hight is 15 feet, and the base 30 feet square?

10. What is the solidity of a pyramid whose altitude is 72, and the sides of whose base are 24, 24, and 40?

11. If the altitude of a cylinder is 7, and the radius of its base 2, what is its whole surface? Its solidity?

12. Prove that a circle described on the hypotenuse of a right-angled triangle as diameter, is equivalent to the sum of the circles described on the other two sides.

13. Prove that if a perpendicular be let fall from the right angle of a right-angled triangle on the hypotenuse, either of the two sides containing the right angle will be a mean proportional between the hypotenuse and the adjacent part of the hypotenuse.

14. If from the top of a mountain 2.006 miles high a vessel can be seen in the horizon 126 miles off, what is the diameter of the earth? (Theo. XIII, B. II).

15. If the earth's diameter be 7912 miles, what is its circumference? Its surface? Its solidity?

16. If the circumference of the moon be 6783 miles, what is its diameter?

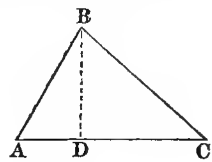
17. Prove that the solidities, S and *s*, of any two spheres are to each other as the cubes of their diameters, D and *d*.

18. The diameter of the sun is 112 times that of the earth; what are the relative solidities of the two bodies?

SEC. XIX.—APPLICATIONS OF ALGEBRA.

For those acquainted with the elements of both branches, a few examples of the application of Algebra to Geometry are subjoined. The method is specially adapted to the solution of that class of problems where certain parts of a figure are given, from which to determine others; but it may also be used for abbreviating long demonstrations of theorems.

1. If a perpendicular be let fall from the vertex of a triangle upon its base, the sum of the parts of the base is to the sum of the other two sides as the difference of the latter is to the difference of the former.



Proof. Let BD be the perpendicular. Now, $BC^2-CD^2=BD^2$ (Cor. 2, Theo. XIX, B. I).

Also, $AB^2-AD^2=BD^2$.

Therefore,

$$BC^2-CD^2=AB^2-AD^2.$$

Transposing, $BC^2-AB^2=CD^2-AD^2$.

Factoring,

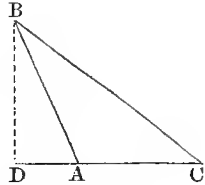
$$(BC+AB)\ (BC-AB)=(CD+AD)\ (CD-AD).$$

Hence, resolving into a proportion, we have

$$CD+AD : CB+AB :: CB-AB : CD-AD.$$

2. The square of the side opposite to any acute angle in a triangle is equal to the sum of the squares of the other sides, minus twice the product of the base by the distance from the acute angle to the foot of the perpendicular let fall from the vertex on the base, or the base produced.

Proof. For the case where the perpendicular falls on the base produced, take the annexed figure; for the other case, take the preceding one.



Now, $AB^2=BD^2+AD^2$ (Theo. XIX, B. I).

But, $AD=CD-AC$ (or, $AC-CD$).

Hence, $AD^2=CD^2+AC^2-2\,CD\times AC$.

Substituting this in the first equation, we have

$$AB^2=BD^2+CD^2+AC^2-2\,CD\times AC.$$

But $BD^2+CD^2=BC^2$; whence,

$$AB^2=BC^2+AC^2-2\,CD\times AC.$$

3. The square of the side opposite an obtuse angle in a triangle is equal to the sum of the squares of the other sides, plus twice the product of the base by the distance from the obtuse angle to the foot of the perpendicular, let fall from the vertex on the base produced.

For proof use the last figure.

4. Given the base and the sum of the two other sides of a right-angled triangle, to find the hypotenuse and perpendicular.

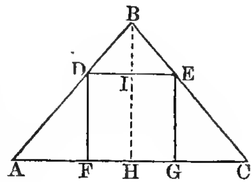
Solution. Represent the base by b and the sum of the other sides by s. Also, represent the perpendicular by x, then the hypotenuse will be $s-x$.

Now, $b^2+x^2=(s-x)^2$ (Theo. XIX, B. I).

Hence, by reduction, we have,

$$x=\frac{s^2-b^2}{2s}.$$

5. Given the base and altitude of a triangle, to find the side of the inscribed square.



Solution. Represent the base AC by b, the altitude BH by h, and the side of the inscribed square by x. Then BI will be represented by $h—x$.

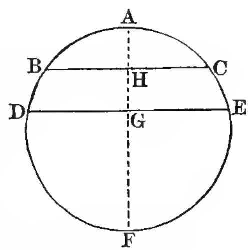
Now, by similarity of triangles,

$$h : b :: h-x : x.$$

Hence, $$hx=bh-bx.$$

And, $$x=\frac{bh}{b+h}.$$

6. If two parallel chords in a circle be 96 and 60, and the distance between them 26, what is the diameter?



Solution. Draw the diameter AF perpendicular to the chords, and it will bisect them (Theo. XXV, B. I). Represent AH by x and GF by y. Here, since we have introduced two unknown quantities, we must have two independent equations.

Now, $x(26+y)=(30)^2$ (Theo. XII, B. II).

And $(x+26)y=(48)^2$.

Finding x and y from these equations, and adding their sum to 26, we shall have the diameter$=100$.

7. Given the diagonal and perimeter of a rectangle, to find the sides.

8. In a triangle, given the base b, the altitude h, and the ratio of the other two sides, as m to n, to find the sides.

Represent the sides by mx and nx.

9. Given the base and altitude of any triangle and the difference of the other two sides, to find the sides.

Let x stand for the half sum of the two sides, and d for their half difference, then the sides will be represented by $x+d$ and $x-d$.

10. Given the base and the other two sides of a triangle, to find the altitude.

11. Given the area of a rectangle inscribed in a given triangle, to find the sides of the rectangle.

12. Given the radius of a circle, to find the side of the inscribed equilateral triangle.

OHIO.

From **HON. ANSON SMYTH,** *late State Supt. of Public Instruction, Ohio.*

McGuffey's *Old* Eclectic Readers I esteemed as among the *very best* works of the kind; but the *New* are certainly a decided improvement upon the Old. I know of no others which I could more earnestly and honestly indorse.

I have examined with care the new editions of Ray's Arithmetics, and am greatly pleased with them. The Primary is certainly a very decided improvement: I have seen *none* so well adapted for a text-book in the elements of Mental Arithmetic. The Intellectual is an *admirable* work. The importance of Mental Arithmetic is now generally appreciated; and I know of no work that embodies so systematic, complete, and thorough a course in this useful branch of study. Ray's Practical Arithmetic *needs* no praise. It is its own commendation.

Pinneo's Series of Grammars I esteem as among the *best* text-books extant, for guiding the learner to a knowledge of the correct use of our language. The definitions are clear and exact; the rules are simple and comprehensive; and the whole plan and arrangement well adapted to achieve their purpose.

INDIANA.

From **HON. M. J. FLETCHER,** *former State Supt. Public Instruction, Ind.*

The public sentiment, as expressed in Indiana by the almost universal use of the Eclectic Educational Series of School Books, embracing McGuffey's New Series of Readers and Speller, Ray's Series of Arithmetics and Algebras, and Pinneo's Series of Grammars, was sufficient in itself to induce the State Board of Education to adopt them.

In addition to this, by careful examination, I am well satisfied that their true intrinsic and relative merit entitles them to such recommendation. They are printed on good, firm, substantial white paper, are durably bound, and of unrivaled cheapness.

ILLINOIS.

From **HON. N. BATEMAN,** *State Supt. of Public Instruction, Illinois.*

No series of books has *ever* obtained as many voices of approval from teachers as McGuffey's Eclectic Readers. Certainly no other Series has been so popular throughout the West. We unhesitatingly say that *we know of no better books*, and should not take the trouble to look for any. The printing is beautiful, the paper very fine, and the binding good: and McGuffey's Readers are proverbially *cheap*.

RAY'S ARITHMETICS have deservedly shared in the popularity of the Eclectic Series. The HIGHER ARITHMETIC is better than any other that we know to be used in this country. RAY'S ALGEBRAS are clear, full, and comprehensive. We advise all who wish to arrange a course of studies, including Algebra, to examine these before choosing.

MINNESOTA.

From **HON. B. F. CRARY,** *former State Supt. of Public Instruction, Minn.*

I have examined McGUFFEY'S NEW ECLECTIC READERS, and have no hesitation in saying that they are *superior* to any similar text-books that have come under my observation. The standard of morals and taste in the Readers is very high, and in their LOW PRICE, and beautiful printing and binding, they distance all competition. I rejoice that a *Western House* has been able to meet the increasing wants of the West in this great field.

IOWA.

From **HON. ORAN FAVILLE,** *State Supt. of Public Instruction, Iowa.*

Having recently reëxamined the ECLECTIC SERIES OF SCHOOL BOOKS, I am fully confirmed in the opinion that they are *the best* Series, on the whole, now in use in the West. Their remarkable popularity, and the continued attachment manifested for them by practical Educators, give evidence both of their intrinsic worth, and of their adaptation to the place designed for them.

Without specifying further, I will say that McGUFFEY'S NEW ECLECTIC SERIES OF READERS, SPELLER, and PRIMARY SCHOOL CHARTS, PINNEO'S SERIES OF GRAMMARS, and RAY'S SERIES OF ARITHMETICS and ALGEBRAS, are *unsurpassed* by any similar Series with which I am acquainted. I therefore recommend their continued use in our State.

MISSOURI.

From **HON. W. B. STARKE,** *former State Supt. of Public Instruction, Mo.*

I have taken much pains to ascertain what are the most approved text-books throughout the country, and after free consultation with leading teachers from different sections of the State, and with their hearty sanction of this course, I recommend the following list of books to be used in the Common Schools of Missouri: McGUFFEY'S NEW SERIES OF READERS, SPEAKERS, and SPELLER, PINNEO'S SERIES OF GRAMMARS, and RAY'S SERIES OF ARITHMETICS and ALGEBRAS.

WISCONSIN.

From **HON. J. L. PICKARD,** *State Supt. of Public Instruction, Wis.*

The books I have recommended below, [McGuffey's New Readers, Ray's Arithmetics, Pinneo's Grammar, and White's Class-Book of Geography,] are such as commend themselves to my judgment. I would advise their adoption in all schools where no uniformity at present exists.

KANSAS.

From **HON. WM. R. GRIFFITH,** *State Supt. of Public Instruction, Kan.*

I recommend McGuffey's New Eclectic Series of Readers, Speakers, and Speller, and Ray's Series of Arithmetics and Algebras to the favorable consideration of the Teachers of our Public Schools. These works possess *real merit*, and I trust they will be approved by the citizens of the State generally. I have spent a week in examining McGuffey's Series, and *I most heartily commend them.*

I have also, after careful examination, concluded to recommend Pinneo's Series of Grammars. I have endeavored to examine the most popular works on the subject of Grammar, as a teacher rather than as a critic, and, in so doing, have been *compelled* to give my preference to Pinneo's. The early introduction of *analysis*, and the abundant *black-board exercises* provided, make Pinneo's Grammars *very practical works.*

VALUABLE TESTIMONY.

From **Rev. BISHOP CLARK, D. D.,** *formerly Editor of the Ladies' Repository.*

I have had frequent occasion, during the past few years, to examine and reëxamine the Eclectic Educational Series. Taken as a whole, they are unquestionably the best issued by any house in America. The popularity enjoyed by the Eclectic Series rests upon the substantial basis of merit.

From **I. W. ANDREWS, D. D.,** *President of Marietta College.*

I have examined carefully McGuffey's Eclectic Readers, and am prepared to speak of them in terms of *unqualified* commendation. They appear to me to combine more excellences than any other readers with which I am acquainted.

The favorable opinion I had formed of them from examination has been confirmed by the use of them in my own family. I was really charmed with them, and so were my children. I do not believe better books for this purpose were ever prepared: *I have never seen any as good*

From **A. J. RICKOFF,** *former Supt. of Public Schools, Cincinnati, Ohio.*

It is now nearly twenty years since I commenced the use of Ray's Arithmetics. During all that time, they have been used either in schools under my supervision or in my own school-rooms. In the mean time, all the prominent American books in this department have come under my examination, and now I have to say that I have found nothing superior to them for the use of either pupil or teacher.

From Prof. **EDWARD BROOKS,** *Prof. of Mathematics, Pennsylvania State Normal School.*

Ray's Algebras are very excellent works; I do not know their superiors. I have used them in this institution for the last eight years with *great satisfaction.*

From **W. W. WOODRUFF,** *Supt. of Public Schools, Chester County, Pa.*

I can say, from my past experience in teaching Pinneo's Grammars, that I regard them as containing the *clearest exhibition* of the principles of our language, that I have met with in the school-room. Other works are more elaborate; others have again more novel and interesting methods of presenting some parts of the subject; but no author has presented English Grammar in a clearer or simpler form than Pinneo.

From **Rev. CYRUS NUTT, D. D.,** *Pres. of Indiana State University.*

I take pleasure in recommending McGuffey's New Eclectic Series of Readers, believing them to be well deserving of their very great popularity, and worthy of universal adoption. Taken in every respect, *they are the best Series of School Readers now published.*

From **G. A. WENTWORTH,** *Prof. of Mathematics, Phillips' Academy, Exeter, New Hampshire.*

Having thoroughly tested Ray's Higher Arithmetic in the recitation room, with a class of some forty pupils, I am prepared to say that *no other* Arithmetic with which I am acquainted has proved so satisfactory. It ought to supplant all other works of the kind in our High Schools and Academies.

From Prof. **G. A. TRACY,** *Brooklyn Academy, Brooklyn, Conn.*

I have used Ray's Arithmetics and Algebras nearly two years, and during my experience as a teacher, I have not found any works on mathematical science, *so well* calculated to give the student a thorough knowledge of the subject, and such discipline of mind, as Prof. Ray's.

ECLECTIC EDUCATIONAL SERIES